Quandle and link-homotopy

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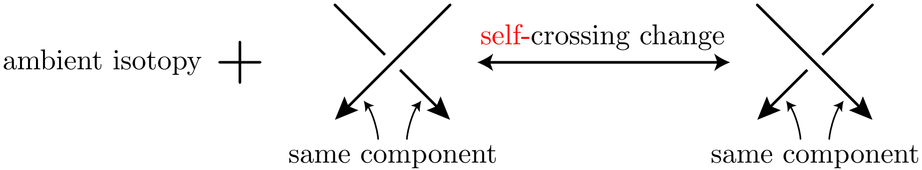
1. Motivation and results

link-homotopy is ...



Rough history

- J. Milnor (1954, 1957)
  - Defined the notion of link-homotopy
  - Defined Milnor invariants ($\overline{\mu}$ invariants)
  - Classified 3-component links up to link-homotopy completely
- J. P. Levine (1988)
  - Enhanced Milnor invariants
  - Classified 4-component links up to link-homotopy completely
- N. Habegger and X. S. Lin (1990)
  - Gave a necessary and sufficient condition for link-homotopic
  - Gave an algorithm judging two links are link-homotopic or not

Motivation

"Classify link-homotopy classes by invariants"

$$\text{numerical invariants} \Rightarrow \begin{cases} \text{easy to compute} \\ \text{easy to compare} \end{cases}$$

This talk

We have a lot of numerical invariants
if we modify the definition of a quandle cocycle invariant slightly.

Definition (quasi-trivial quandle)

X : quandle

X is quasi-trivial

$\overset{\text{def}}{\Leftrightarrow}$ ${}^\forall x \in X, {}^\forall \varphi \in \mathrm{Inn}(X),\ x * \varphi(x) = x.$

Theorem

X : quasi-trivial quandle, A : abelian group

$\theta : X \times X \to A$ 2-cocycle satisfying the condition

(C3) ${}^\forall x \in X, {}^\forall \varphi \in \mathrm{Inn}(X),\ \theta(x, \varphi(x)) = 0$

The quandle cocycle invariant w.r.t. θ is a link-homotopy invariant.

This quandle cocycle invariant knows
which components are trivial up to link-homotopy.

Talk plan

1. Motivation and results
2. Short review of quandle cocycle invariant
3. How do we ensure the invariance?
4. backstage

2. Short review of quandle cocycle invariant

Definition (quandle)

X : set ($\neq \emptyset$)

$* : X \times X \to X$ binary operation

$(X, *)$: quandle

$\overset{\text{def}}{\Leftrightarrow}$ $*$ satisfies the following axioms:

(Q1) ${}^\forall x \in X$, $x * x = x$.

(Q2) ${}^\forall x \in X$, $*x : X \to X$ $(\bullet \mapsto \bullet * x)$ is bijective.

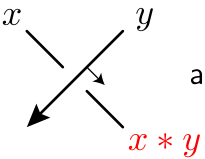
(Q3) ${}^\forall x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

Definition (coloring)

X : quandle

D : (oriented) link diagram

$\mathscr{C} : \{\text{arcs of } D\} \to X$ is an X-coloring of D

$\overset{\text{def}}{\Leftrightarrow}$ $\mathscr{C}$ satisfies the condition  at each crossing.

Fact

$\sharp\{X\text{-colorings of a diagram}\}$

is invariant under the Reidemeister moves.

Definition (2-cocycle)

X : quandle

A : abelian group

$\theta : X \times X \to A$ is a 2-cocycle of X

$\overset{\text{def}}{\Leftrightarrow}$ θ satisfies the following conditions:

(C1) ${}^{\forall}x \in X$, $\theta(x, x) = 0$.

(C2) ${}^{\forall}x, y, z \in X$,

$$\theta(x, y) + \theta(x * y, z) = \theta(x, z) + \theta(x * z, y * z).$$

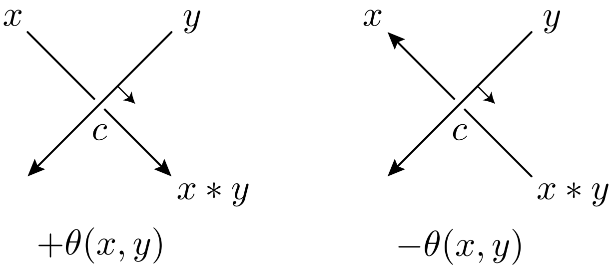
Definition (weight)

$\mathscr{C}$: X-coloring of a diagram

$\theta : X \times X \to A$ 2-cocycle

The weight of $\mathscr{C}$ w.r.t. θ is the sum

$$\sum_{c\,:\,\text{crossing}} \text{sign}(c) \cdot \theta(x, y) \ \in A.$$



Theorem (J. S. Carter et al. 2003)

X : quandle

A : abelian group

$\theta : X \times X \to A$ 2-cocycle

For each link L, the multiset

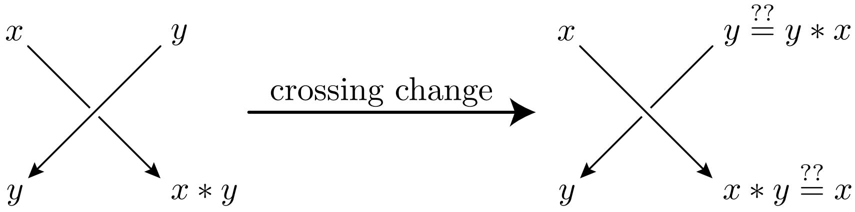
$$\{(\text{weight of } \mathscr{C} \text{ w.r.t. } \theta) \in A \mid \mathscr{C} : X\text{-coloring of a diagram of } L\}$$

is invariant under the Reidemeister moves.

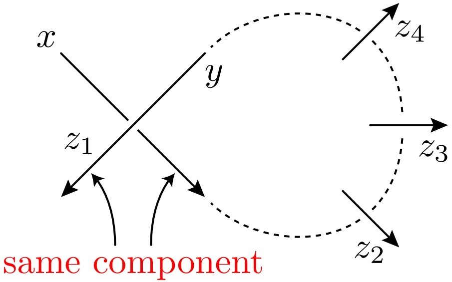
We call this multiset the quandle cocycle invariant of L w.r.t. θ.

3. How do we ensure the invariance?

Investigation for colorings



The set of colorings is NOT invariant under crossing changes, in general.



$$
\begin{aligned}
y &= (((x * z_1) * z_2) * z_3) * z_4 \\
&= \varphi(x) \quad (\varphi \in \mathrm{Inn}(X)).
\end{aligned}
$$

- $\mathrm{Aut}(X) := \{\varphi : X \to X \text{ auto.}\}$: automorphism group of X
- $\mathrm{Inn}(X) := \langle * x : X \to X \ (x \in X) \rangle \triangleleft \mathrm{Aut}(X)$

  : inner automorphism group of X

Definition (quasi-trivial quandle)

X : quandle

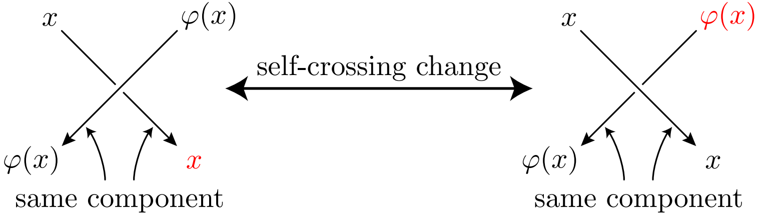
X is quasi-trivial

$\overset{\text{def}}{\Leftrightarrow}$ ${}^{\forall}x \in X$, ${}^{\forall}\varphi \in \mathrm{Inn}(X)$, $x * \varphi(x) = x$.

Lemma

X : quasi-trivial quandle

$\sharp\{X\text{-colorings of a diagram}\}$ is invariant up to link-homotopy.

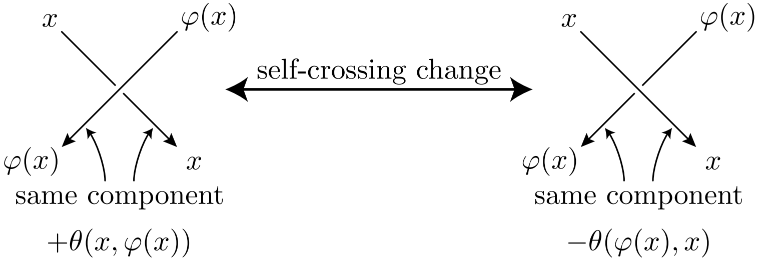


Investigation for weights

X : quasi-trivial quandle

$\mathscr{C}$: X-coloring of a diagram

$\theta : X \times X \to A$ 2-cocycle



Consider the following condition:

(C3) $\forall x \in X, \forall \varphi \in \mathrm{Inn}(X), \ \theta(x, \varphi(x)) = 0$

Theorem

X : quasi-trivial quandle

A : abelian group

$\theta : X \times X \to A$ 2-cocycle satisfying the condition (C3)

(C3) $^{\forall}x \in X$, $^{\forall}\varphi \in \mathrm{Inn}(X)$, $\theta(x, \varphi(x)) = 0$

The quandle cocycle invariant w.r.t. θ is a link-homotopy invariant.

Example (non-triviality of the Borromean rings)

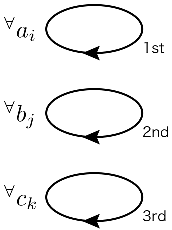
X : quasi-trivial quandle

$*$	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	c_1	c_2	c_3	c_4
a_1	a_1	a_1	a_1	a_1	a_2	a_2	a_2	a_2	a_3	a_3	a_3	a_3
a_2	a_2	a_2	a_2	a_2	a_1	a_1	a_1	a_1	a_4	a_4	a_4	a_4
a_3	a_3	a_3	a_3	a_3	a_4	a_4	a_4	a_4	a_1	a_1	a_1	a_1
a_4	a_4	a_4	a_4	a_4	a_3	a_3	a_3	a_3	a_2	a_2	a_2	a_2
b_1	b_3	b_3	b_3	b_3	b_1	b_1	b_1	b_1	b_2	b_2	b_2	b_2
b_2	b_4	b_4	b_4	b_4	b_2	b_2	b_2	b_2	b_1	b_1	b_1	b_1
b_3	b_1	b_1	b_1	b_1	b_3	b_3	b_3	b_3	b_4	b_4	b_4	b_4
b_4	b_2	b_2	b_2	b_2	b_4	b_4	b_4	b_4	b_3	b_3	b_3	b_3
c_1	c_2	c_2	c_2	c_2	c_3	c_3	c_3	c_3	c_1	c_1	c_1	c_1
c_2	c_1	c_1	c_1	c_1	c_4	c_4	c_4	c_4	c_2	c_2	c_2	c_2
c_3	c_4	c_4	c_4	c_4	c_1	c_1	c_1	c_1	c_3	c_3	c_3	c_3
c_4	c_3	c_3	c_3	c_3	c_2	c_2	c_2	c_2	c_4	c_4	c_4	c_4

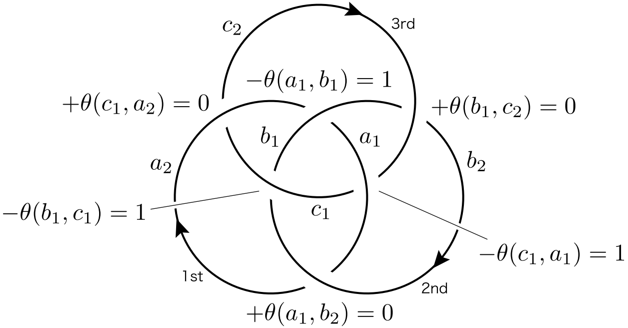
$\theta : X \times X \to \mathbb{Z}_2$ 2-cocycle satisfying the condition (C3)

θ	a_1	a_2	a_3	a_4	b_1	b_2	b_3	b_4	c_1	c_2	c_3	c_4
a_1	0	0	0	0	1	0	1	0	1	1	0	0
a_2	0	0	0	0	0	1	0	1	0	0	1	1
a_3	0	0	0	0	1	0	1	0	1	1	0	0
a_4	0	0	0	0	0	1	0	1	0	0	1	1
b_1	1	1	0	0	0	0	0	0	1	0	1	0
b_2	0	0	1	1	0	0	0	0	0	1	0	1
b_3	1	1	0	0	0	0	0	0	1	0	1	0
b_4	0	0	1	1	0	0	0	0	0	1	0	1
c_1	1	0	1	0	1	1	0	0	0	0	0	0
c_2	0	1	0	1	0	0	1	1	0	0	0	0
c_3	1	0	1	0	1	1	0	0	0	0	0	0
c_4	0	1	0	1	0	0	1	1	0	0	0	0

L_1



L_2 (Borromean rings)



weight $= 0$ weight $= 1$

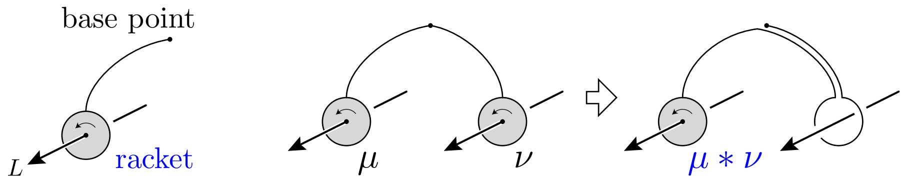
$$\therefore \ L_1 \not\sim L_2.$$

Remark

$$\sharp\{X\text{-colorings of } L_1\} = \sharp\{X\text{-colorings of } L_2\}.$$

4. Backstage

L : link



$Q(L) := \{\text{rackets of } L\}/\text{homotopy}.$

$(Q(L), *)$: knot quandle of L

X : quandle

$\mathscr{C}$: X-coloring of a diagram of L $\overset{1:1}{\longleftrightarrow}$ $f_{\mathscr{C}} : Q(L) \to X$ homo.

$L = K_1 \cup K_2 \cup \cdots \cup K_n$

$[K_i] \in H_2^Q(Q(L);\mathbb{Z})$: fundamental class a.w. K_i

X : quandle, $f_{\mathscr{C}} : Q(L) \to X$ homo. ($\leftrightarrow \mathscr{C}$: X-coloring of L)

$\theta : X \times X \to A$ 2-cocycle, i.e., $\theta \in Z_Q^2(X;A)$

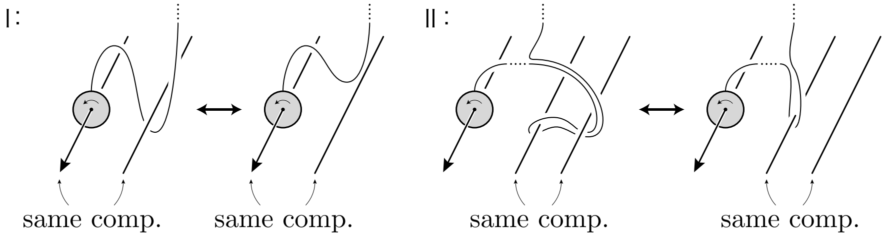
$$(\text{weight of } \mathscr{C} \text{ w.r.t. } \theta) = \sum_{i=1}^{n} \langle [\theta], f_{\mathscr{C}}^*([K_i]) \rangle$$

Theorem (M. Eisermann 2003)

$L = K_1 \cup \cdots \cup K_m \cup K_{m+1} \cup \cdots \cup K_n$

$K_1, \ldots, K_m$: non-trivial, $K_{m+1}, \ldots, K_n$: trivial

$$H_2^Q(Q(L);\mathbb{Z}) = \mathrm{span}_{\mathbb{Z}}\{[K_1], \ldots, [K_m]\} \cong \mathbb{Z}^m.$$



$RQ(L) := Q(L)/(\text{the above moves})$

$(RQ(L), *)$: reduced knot quandle of L (J. R. Hughes 2011)

Theorem (J. R. Hughes 2011)

$RQ(L)$ is invariant up to link-homotopy.

X : quasi-trivial quandle

$\mathscr{C}$: X-coloring of a diagram of L $\overset{1:1}{\longleftrightarrow}$ $f_{\mathscr{C}} : RQ(L) \to X$ homo.

X : quasi-trivial quandle

A : abelian group

$H_n^{QT}(X;A)$ ($H^n_{QT}(X;A)$) : modified quandle (co)homology group

$\rightsquigarrow [K_i] \in H_2^{QT}(RQ(L);\mathbb{Z})$: fundamental class a.w. K_i
(well-defined up to link-homotopy)

Remark

$\theta : X \times X \to A$ 2-cocycle

θ satisfies the condition (C3) $\Leftrightarrow$ $\theta \in Z^2_{QT}(X;A)$.

X : quasi-trivial quandle, $f_{\mathscr{C}} : RQ(L) \to X$ homo. $(\leftrightarrow \mathscr{C})$

$\theta : X \times X \to A$ 2-cocycle satisfying (C3), i.e., $\theta \in Z^2_{QT}(X; A)$

$$\text{(weight of } \mathscr{C} \text{ w.r.t. } \theta) = \sum_{i=1}^{n} \langle [\theta], f^*_{\mathscr{C}}([K_i]) \rangle$$

Theorem

$L = K_1 \cup \cdots \cup K_m \cup K_{m+1} \cup \cdots \cup K_n$

$K_1, \ldots, K_m$: non-trivial up to link-homotopy

$K_{m+1}, \ldots, K_n$: trivial up to link-homotopy

Then $H^{QT}_2(RQ(L); \mathbb{Z})$ is generated by $[K_1], [K_2], \ldots, [K_m]$.