A property of normalized arrow polynomials of checkerboard colorable virtual links

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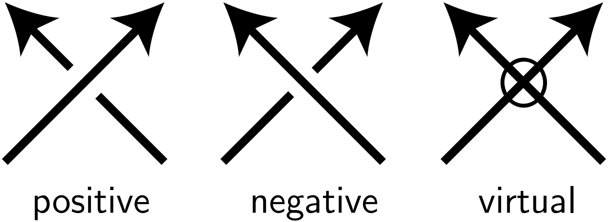
Definition

virtual link diagram

Def

A **virtual link diagram** is a link diagram which may have virtual crossings.

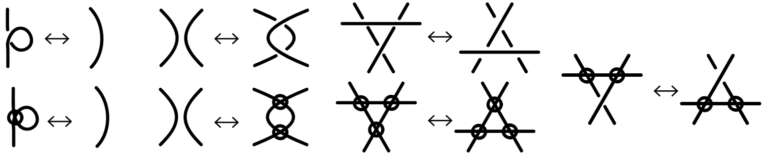


Definition

virtual link

Def

A **virtual link** is the equivalence class of virtual link diagram under the generalized Reidemeister moves.

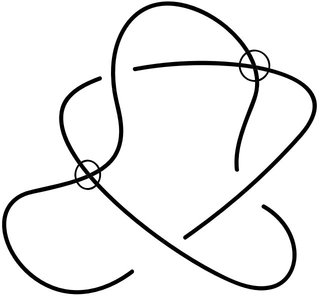


Generalized Reidmeister moves

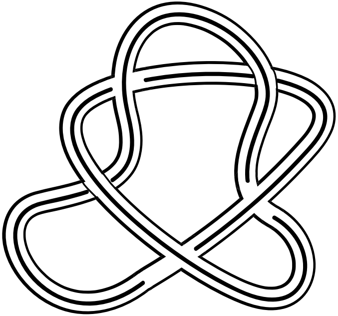
Definition

abstract link diagram

Def(abstract link diagram(ALD))



virtual link diagram D



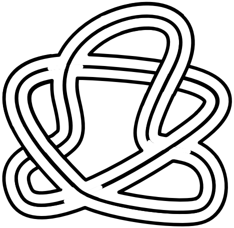
ALD associated with D

Definition

checkerboard coloring

Def

A **checkerboard coloring** of ALD is a coloring of regions by black and white, such that adjacent regions have different colorings.



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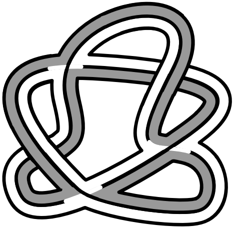

Definition

checkerboard colorable

Def

D : a virtual link diagram.

D is **checkerboard colorable**

$\Longleftrightarrow$ an ALD associated with D is checkerboard colorable.

Definition

checkerboard colorable

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D : a virtual link diagram.

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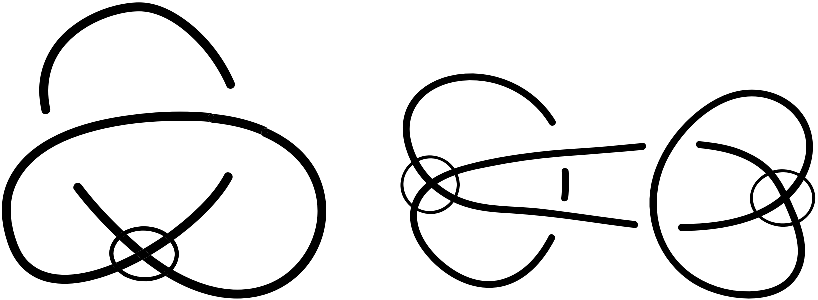
$\Longleftrightarrow$ an ALD associated with D is checkerboard colorable.

Fact

Any classical link diagram is checkerboard colorable.

Example

EX



Definition

checkerboard colorable

Def

L : a virtual link.

L is **checkerboard colorable**

$\Longleftrightarrow$ L has a checkerboard colorable virtual link diagram.

We show that a certain virtual link is not checkerboard colorable by using invariants for virtual links.

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In this talk, a virtual link is always oriented.

f-polynomial

Fact[Kauffman '99]f-polynomial

f: {virtual link}$\longrightarrow Z[A, A^{-1}]$

f-polynomial

Fact[Kauffman '99]f-polynomial
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Thm[N.Kamada '02]
L : a virtual link with n-components.
$f(L)$: a f-polynomial of L.
EXP$(f(L))$: the set of integers appearing as exponents of $f(L)$.
ex) $f(L) = 2A^6 + A^4 - A^{-2}$
EXP$(f(L)) = \{6, 4, -2\}$

f-polynomial

Fact[Kauffman '99]f-polynomial

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Thm[N.Kamada '02]

L : a virtual link with n-components.

$f(L)$: a f-polynomial of L.

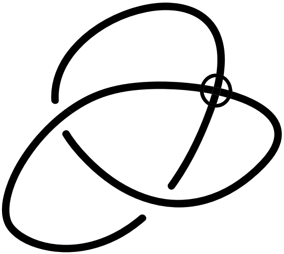
L is checkerboard colorable

$$\Longrightarrow \mathrm{EXP}(f(L)) \subset \begin{cases} 4Z & n \text{ is odd} \\ 4Z+2 & n \text{ is even} \end{cases}$$

Example

f-polynomial

EX



$$f(L) = A^{-4} + A^{-6} - A^{-10}$$

$$\mathrm{EXP}(f(L)) = \{-4, -6, -10\}$$

Definition

normalized arrow polynomial

L : a virtual link with n-components.
D : a virtual link diagram of L.
A state S of D is obtained by choosing A-splice or B-splice for each crossing in D.

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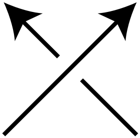

positive

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normalized arrow polynomial

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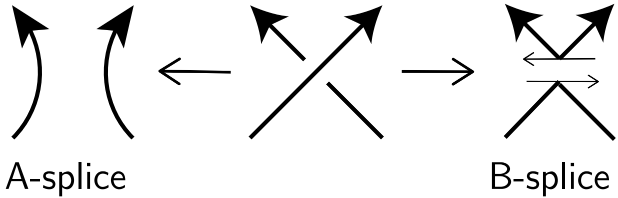
A state S of D is obtained by choosing A-splice or B-splice for each crossing in D.



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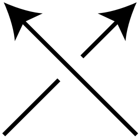

negative

Definition

normalized arrow polynomial

L : a virtual link with n-components.

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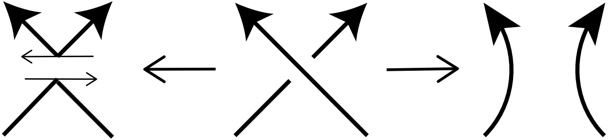
A state S of D is obtained by choosing A-splice or B-splice for each crossing in D.



A-splice B-splice

Definition

normalized arrow polynomial

s : a state of D.
$\sigma(s) = \#\{\text{A-splice}\} - \#\{\text{B-splice}\}$.
$l(s)$ is the number of components of s.
An arrow polynomial of D is defined by

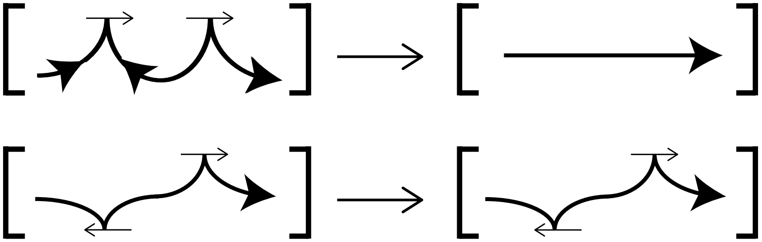
$$\mathcal{A}(D) := \sum_{s} A^{\sigma(s)}(-A^2 - A^{-2})^{l(s)-1}\mathcal{K}(s).$$

$\mathcal{K}(s)$ is a product of extra variables $K_1, K_2 \cdots$ associated with the state s.

Definition

normalized arrow polynomial

State Reduction



Definition

normalized arrow polynomial

After State Reducation, s consists of some loops with alterneing arrows.

α_m : the loop with $2m$ alternating arrows.

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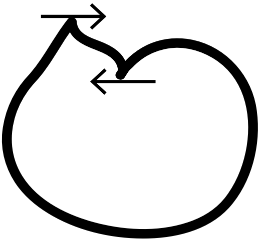
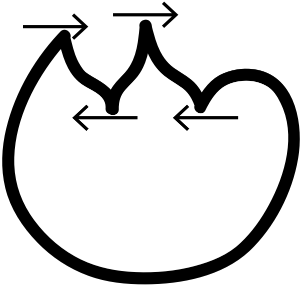


α_1



α_2

Definition

normalized arrow polynomial

$\{l_1, \ldots, l_p\}$: the set of the loops of s.

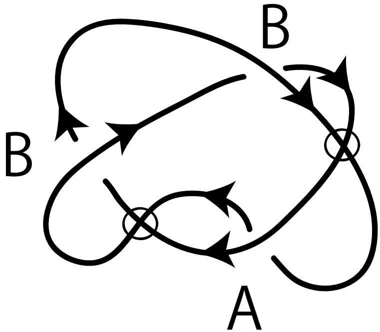
$$\mathcal{K}(s) := K_{a(l_1)} K_{a(l_2)} \ldots K_{a(l_p)}$$

If l_i is α_m, then $a(l_i)$:=m. K_0:=1.

Definition

normalized arrow polynomial

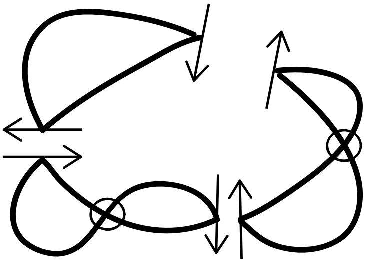
EX



Definition

normalized arrow polynomial

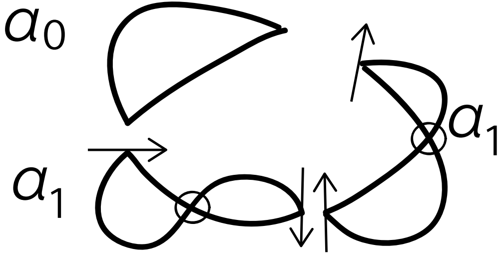
EX



Definition

normalized arrow polynomial

EX



$$\mathcal{K}(s) = K_0 K_1 K_1 = 1 K_1 K_1 = K_1^2.$$

Definition

normalized arrow polynomial

Thm[Kauffman '10]
$\mathcal{A}(D) := \sum_s A^{\sigma(s)}(-A^2 - A^{-2})^{l(s)-1}\mathcal{K}(s)$.
$\omega(D)$: the writhe of D.
A normalized arrow polynomial of D is defined by

$$\mathcal{W}(D) := (-A^{-3})^{\omega(D)}\mathcal{A}(D).$$

$\mathcal{W}(D)$ is an invariant for virtual links.

normalized arrow polynomial

I introduce two methods of detecting checkerboard colorability of virtual links by using a normalized arrow polynomial.

normalized arrow polynomial

Thm1

L : a virtual link with n-components.

$\mathcal{W}(L)$: a normalized arrow polynomial of L.

L is checkerboard colorable

$$\Longrightarrow \mathrm{EXP}(\mathcal{W}(L)) \subset \begin{cases} 4Z & n \text{ is odd} \\ 4Z+2 & n \text{ is even} \end{cases}$$

normalized arrow polynomial

$K_{n_1}^{m_1} K_{n_2}^{m_2} \ldots K_{n_t}^{m_t}$ is good.

normalized arrow polynomial

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$$\iff P_i = \begin{cases} n_i & m_i \text{ is odd} \\ 0 & m_i \text{ is even} \end{cases},$$

$K_{P_1} K_{P_2} \ldots K_{P_t}$ is good.

normalized arrow polynomial

$K_{n_1}^{m_1} K_{n_2}^{m_2} \dots K_{n_t}^{m_t}$ is good.

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$K_{P_1} K_{P_2} \dots K_{P_t}$ is good.

$$\overset{def}{\iff} \exists a_t: \text{ sequence s.t. }$$

$$\sum_{i=1}^{t} a_i P_i = 0, a_i = \pm 1.$$

normalized arrow polynomial

$$K_{n_i}^{m_i} \Longrightarrow \begin{cases} K_{n_i} & m_i \text{ is odd} \\ K_0 = 1 & m_i \text{ is even} \end{cases}$$

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※ If $m_1, m_2, \ldots, m_t$ are even, then this term is good.

normalized arrow polynomial

EX

$K_1 K_2 K_3 K_4^2$ is good.

$\because)$ $K_1 K_2 K_3 K_0 = K_1 K_2 K_3$ is good.

$K_1 K_2 K_3^2$ is not good.

$\because)$ $K_1 K_2 K_0 = K_1 K_2$ is not good.

normalized arrow polynomial

EX

$K_1K_2K_3K_4^2$ is good.

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$K_1K_2K_3^2$ is not good.

$\because$) $K_1K_2K_0 = K_1K_2$ is not good.

K_1^2 is good.

K_2 is not good.

normalized arrow polynomial

Thm2

L : a virtual link.
$\mathcal{W}(L)$: a normalized arrow polynomial of L.
If L is checkerboard colorable, then any term of $\mathcal{W}(L)$ is good.

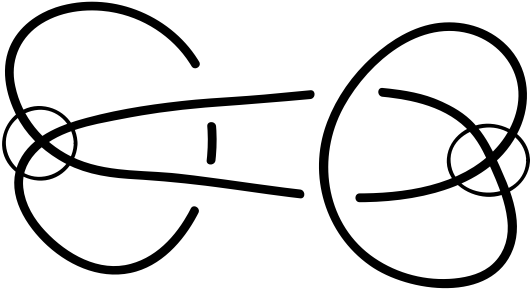
normalized arrow polynomial

Thm

The method of detecting checkerboard colorability of virtual links by using Thm2 is stronger than that by using Thm1.

normalized arrow polynomial

EX



$$\mathcal{W}(L) = 1 + A^4 + A^{-4} - (2 + A^4 + A^{-4})K_1^2 + 2K_2.$$