The optimistic limit of colored Jones polynomial

Jinseok Cho
(joint work with Jun Murakami)

Korea Institute for Advanced Study

January 11, 2012

Hyperbolic knot

Definition (Upper half space model of $\mathbb{H}^3$)

$$\mathbb{H}^3 := \{(z,t) | z \in \mathbb{C}, t > 0\}$$

with the metric

$$ds^2 := \frac{dz^2 + dt^2}{t^2}$$

is called the *hyperbolic 3-space*. $\mathbb{H}^3$ is a Riemannian 3-manifold with the constant sectional curvature -1.

Definition

A Riemannian manifold M is called *hyperbolic* if it has a complete metric with the constant sectional curvature -1.

Hyperbolic knot

Definition

A knot K is called *hyperbolic* if the knot complement $\mathbb{S}^3 \backslash K$ is hyperbolic.

For example, 4_1 and 5_2 knots are hyperbolic knots.
Thurston proved that almost all knots are hyperbolic.

Theorem (Mostow Rigidity)

Let M and N be n-dimensional hyperbolic manifolds with $n \geq 3$. If M is homotopic to N, then M is isomorphic to N.

Mostow rigidity implies the hyperbolic structure of a hyperbolic knot is unique. Therefore, we can use geometric quantities of a hyperbolic knot as topological invariants.

Volume Conjecture

Conjecture (Volume conjecture)

For a hyperbolic knot K,

$$\mathrm{vol}(K) = 2\pi \lim_{N\to\infty} \frac{\log |J_N(K; \exp(2\pi i/N))|}{N},$$

where $\mathrm{vol}(K)$ is the volume of the hyperbolic manifold $\mathbb{S}^3 \backslash K$ and $J_N(K;q)$ is the N-th colored Jones polynomial of K.

Conjecture (Complex volume conjecture)

$$\mathrm{vol}(K) + i\,\mathrm{cs}(K) \equiv 2\pi \lim_{N\to\infty} \frac{\log J_N(K; \exp(2\pi i/N))}{N} \pmod{i\pi^2},$$

where $\mathrm{cs}(K)$ is the Chern-Simons invariant of the hyperbolic manifold $\mathbb{S}^3 \backslash K$.

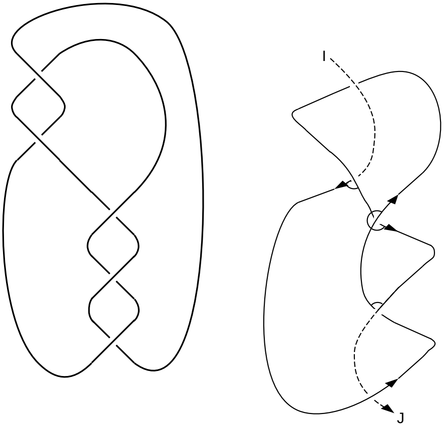
We call $\mathrm{vol}(K) + i\,\mathrm{cs}(K)$ *the complex volume of K.*

Why optimistic limit?

To calculate $\mathrm{vol}(K)$, we usually triangulate $\mathbb{S}^3 \backslash K$ and add the volumes of all the tetrahedra. On the other hand, if we could calculate the limit of the colored Jones polynomial easily, it would give a convenient way to calculate the volume of the hyperbolic knot.
Although the actual calculation of the limit is extremely hard, there is a simple calculating method, called *the optimistic limit*, which we expect to give the actual limit. We introduce this method here.

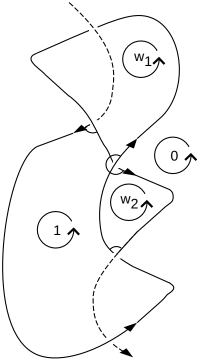
1st step : drawing oriented (1,1) tangle

We assume the knot diagram does not have any reducible crossings.



(a) 5_2 knot (b) Graph G

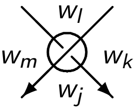
2nd step : defining potential function



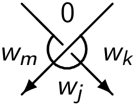
Note that $\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t} dt$ is the dilogarithm function.
For each vertex of G, we assign the following function according to the types of the vertex and the shapes of the arcs.

2nd step : defining potential function

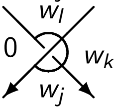
For positive crossings :



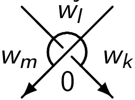
$$: \mathrm{Li}_2\left(\frac{w_l}{w_m}\right) + \mathrm{Li}_2\left(\frac{w_l}{w_k}\right) - \mathrm{Li}_2\left(\frac{w_j w_l}{w_k w_m}\right) - \mathrm{Li}_2\left(\frac{w_m}{w_j}\right) - \mathrm{Li}_2\left(\frac{w_k}{w_j}\right) + \frac{\pi^2}{6} - \log\frac{w_m}{w_j}\log\frac{w_k}{w_j}$$



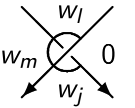
$$: -\mathrm{Li}_2\left(\frac{w_m}{w_j}\right) - \mathrm{Li}_2\left(\frac{w_k}{w_j}\right) + \frac{\pi^2}{6} - \log\frac{w_m}{w_j}\log\frac{w_k}{w_j}$$



$$: \mathrm{Li}_2\left(\frac{w_l}{w_k}\right) + \mathrm{Li}_2\left(\frac{w_j}{w_k}\right) - \frac{\pi^2}{6} + \log\frac{w_k}{w_l}\log\frac{w_k}{w_j}$$



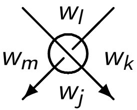
$$: -\mathrm{Li}_2\left(\frac{w_m}{w_l}\right) - \mathrm{Li}_2\left(\frac{w_k}{w_l}\right) + \frac{\pi^2}{6} - \log\frac{w_m}{w_l}\log\frac{w_k}{w_l}$$



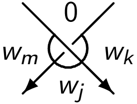
$$: \mathrm{Li}_2\left(\frac{w_l}{w_m}\right) + \mathrm{Li}_2\left(\frac{w_j}{w_m}\right) - \frac{\pi^2}{6} + \log\frac{w_m}{w_l}\log\frac{w_m}{w_j}$$

2nd step : defining potential function

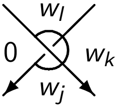
For negative crossings :



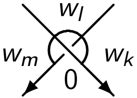
$: \; -\mathrm{Li}_2(\frac{w_l}{w_m}) - \mathrm{Li}_2(\frac{w_l}{w_k}) + \mathrm{Li}_2(\frac{w_j w_l}{w_k w_m}) + \mathrm{Li}_2(\frac{w_m}{w_j}) + \mathrm{Li}_2(\frac{w_k}{w_j})$
$-\frac{\pi^2}{6} + \log\frac{w_j}{w_m}\log\frac{w_j}{w_k}$



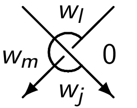
$: \; \mathrm{Li}_2(\frac{w_m}{w_j}) + \mathrm{Li}_2(\frac{w_k}{w_j}) - \frac{\pi^2}{6} + \log\frac{w_j}{w_m}\log\frac{w_j}{w_k}$



$: \; -\mathrm{Li}_2(\frac{w_l}{w_k}) - \mathrm{Li}_2(\frac{w_j}{w_k}) + \frac{\pi^2}{6} - \log\frac{w_l}{w_k}\log\frac{w_j}{w_k}$



$: \; \mathrm{Li}_2(\frac{w_m}{w_l}) + \mathrm{Li}_2(\frac{w_k}{w_l}) - \frac{\pi^2}{6} + \log\frac{w_l}{w_m}\log\frac{w_l}{w_k}$

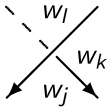


$: \; -\mathrm{Li}_2(\frac{w_l}{w_m}) - \mathrm{Li}_2(\frac{w_j}{w_m}) + \frac{\pi^2}{6} - \log\frac{w_l}{w_m}\log\frac{w_j}{w_m}$

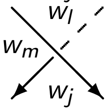
2nd step : defining potential function

For the end points of I and J, we do not concern the arcs around the points.

For the end point of I :

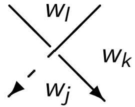


: $-\mathrm{Li}_2(\frac{w_k}{w_l}) + \mathrm{Li}_2(\frac{w_j}{w_l})$

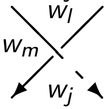


: $\mathrm{Li}_2(\frac{w_m}{w_l}) - \mathrm{Li}_2(\frac{w_j}{w_l})$

For the end point of J :



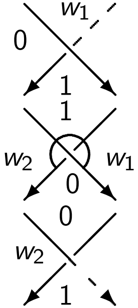
: $-\mathrm{Li}_2(\frac{w_k}{w_j}) + \mathrm{Li}_2(\frac{w_l}{w_j})$



: $\mathrm{Li}_2(\frac{w_m}{w_j}) - \mathrm{Li}_2(\frac{w_l}{w_j})$

2nd step : defining potential function

For example, the potential function $W(w_1, w_2)$ of the 5_2 knot is as follows:



$: \ -\mathrm{Li}_2(\frac{1}{w_1})$

$: \ \mathrm{Li}_2(w_2) + \mathrm{Li}_2(w_1) - \frac{\pi^2}{6} + \log \frac{1}{w_2} \log \frac{1}{w_1}$

$: \ \mathrm{Li}_2(w_2)$

$$W(w_1, w_2) = \mathrm{Li}_2(w_1) - \mathrm{Li}_2(\frac{1}{w_1}) + 2\mathrm{Li}_2(w_2) + \log \frac{1}{w_1} \log \frac{1}{w_2} - \frac{\pi^2}{6}.$$

3rd step : finding essential solutions

For the potential function $W(w_1, \ldots, w_n)$, we define *the hyperbolicity equation*

$$\mathcal{H} := \left\{ \exp\left(w_k \frac{\partial W(w_1, \ldots, w_n)}{\partial w_k} \right) = 1 \,\middle|\, k = 1, \ldots, n \right\}.$$

For example, the hyperbolicity equation of the 5_2 knot is

$$\mathcal{H} = \left\{ \frac{w_2}{(1 - w_1)(1 - \frac{1}{w1})} = 1, \quad \frac{w_1}{(1 - w_2)^2} = 1 \right\}.$$

3rd step : finding essential solutions

Among the solutions of $\mathcal{H}$, we only need the solutions satisfying that none of the variables inside the dilogarithm functions $\mathrm{Li}_2(*)$ of the potential function becomes 0, 1, or ∞. We call these solutions *essential solutions*. For example, the hyperbolicity equation of the 5_2 knot has three essential solutions:

$$\begin{aligned}
(w_1^{(0)}, w_2^{(0)}) &:= (0.1226... - i\,0.7449..., 1.6624... - i\,0.5629...),\\
(\overline{w}_1^{(0)}, \overline{w}_2^{(0)}) &= (0.1226... + i\,0.7449..., 1.6624... + i\,0.5629...),\\
(w_1^{(1)}, w_2^{(1)}) &:= (1.7549..., -0.3247...).
\end{aligned}$$

3rd step : finding essential solutions

In gereral, an essential solution $\mathbf{w}$ of the hyperbolicity equation $\mathcal{H}$ of a knot K induces a representation

$$\rho_{\mathbf{w}} \;:\; \pi_1(K) \longrightarrow \mathrm{Isom}^+(\mathbb{H}^3).$$

Furthermore, we can define the complex volume $\mathrm{vol}(\rho_{\mathbf{w}}) + i\,\mathrm{cs}(\rho_{\mathbf{w}})$ of the representation.

For some solution $\mathbf{w}^{(\mathbf{0})}$ of $\mathcal{H}$, if $\rho_{\mathbf{w}^{(\mathbf{0})}}$ is discrete and faithful, then it induces the hyperbolic structure to $\mathbb{S}^3 \backslash K$. In this case, we call $\rho_{\mathbf{w}^{(\mathbf{0})}}$ and $\mathbf{w}^{(\mathbf{0})}$ *the geometric representation* and *the geometric solution* respectively.

By definition, $\mathrm{vol}(\rho_{\mathbf{w}^{(\mathbf{0})}}) = \mathrm{vol}(K)$ and $\mathrm{cs}(\rho_{\mathbf{w}^{(\mathbf{0})}}) \equiv \mathrm{cs}(K) \pmod{\pi^2}$.

It is a well-known fact that if there exists an essential solution of $\mathcal{H}$, the geometric solution exists in the essential solutions of $\mathcal{H}$.

We assume the existence of an essential solution of $\mathcal{H}$. (If not, we change the diagram and use another potential function.)

Final step : obtaining the complex volume by $i\,W_0$

For the potential function $W(w_1, \dots, w_n)$, we define

$$W_0(w_1, \dots, w_n) := W(w_1, \dots, w_n) - \sum_{k=1}^{n} w_k \frac{\partial W(w_1, \dots, w_n)}{\partial w_k} \log w_k .$$

We evaluate $i\,W_0$ at all essential solutions $\mathbf{w}$ of $\mathcal{H}$. Then the evaluation becomes

$$i\,W_0(\mathbf{w}) \equiv \mathrm{vol}(\rho_{\mathbf{w}}) + i\,\mathrm{cs}(\rho_{\mathbf{w}}) \pmod{i\pi^2}.$$

For example, the evaluations of the 5_2 knot are

$$\begin{aligned}
i\,W_0(w_1^{(0)}, w_2^{(0)}) &= \quad 2.8281... \quad +i\,3.0241...,\\
i\,W_0(\overline{w}_1^{(0)}, \overline{w}_2^{(0)}) &= -2.8281... \quad +i\,3.0241...,\\
i\,W_0(w_1^{(1)}, w_2^{(1)}) &= \quad 0 \quad -i\,1.1135....
\end{aligned}$$

Final step : obtaining the complex volume by $i\,W_0$

For any representation $\rho : \pi_1(K) \to \mathrm{Isom}^+(\mathbb{H}^3)$, it is well-known that

$$\mathrm{vol}(\rho) \leq \mathrm{vol}(K). \tag{1}$$

Furthermore, Gromov-Thurston-Goldman rigidity implies that the equality of (1) holds if and only if ρ is the geometric representation.
Therefore, after confirming the existence of an essential solution of $\mathcal{H}$, if we pick the solution $\mathbf{w}^{(0)}$ of $\mathcal{H}$ which gives the maximum real part of $i\,W_0(\mathbf{w}^{(0)})$, the solution becomes the geometric solution and $i\,W_0(\mathbf{w}^{(0)})$ becomes the complex volume of the knot K.
For example, the complex volume of the 5_2 knot is

$$\begin{aligned}
i\,W_0(w_1^{(0)}, w_2^{(0)}) &= \quad 2.8281\ldots \quad + i\,3.0241\ldots \equiv \mathrm{vol}(5_2) + i\,\mathrm{cs}(5_2),\\
i\,W_0(\overline{w}_1^{(0)}, \overline{w}_2^{(0)}) &= -2.8281\ldots \quad + i\,3.0241\ldots,\\
i\,W_0(w_1^{(1)}, w_2^{(1)}) &= \qquad 0 \qquad\quad - i\,1.1135\ldots.
\end{aligned}$$

Thank you for listening!