A minor-closed class on graph 2-braid groups

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January 11, 2012

Definitions

Let Γ be a finite graph as a CW-complex.

- The *configuration space* of n points in Γ is

$$C_n\Gamma = \{(x_1, \cdots, x_n) \in \Gamma^n | x_i \neq x_j \text{ if } i \neq j\}$$

- The *unordered configuration space* of n points in Γ is

$$UC_n\Gamma = \{\{x_1, \cdots, x_n\} \subset \Gamma | x_i \neq x_j \text{ if } i \neq j\} = C_n\Gamma / S_n.$$

- The *n-braid group* over Γ is $B_n\Gamma = \pi_1(UC_n\Gamma)$.
- *A right-angled Artin group* (**RAAG**) is a group having a presentation whose every relator is commutators of two generators.
- *A n-nucleus* is a minimal graph Γ such that $B_n\Gamma$ is not a RAAG.

Historical facts

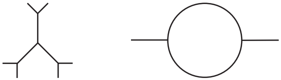
1999 Ghrist conjectured that $PB_n\Gamma$ is a RAAG.

2000 Abrams revised the conjecture so that it holds only for planar graphs.

2006 Sabalka and Farley proved that B_nT is a RAAG for a tree T iff $n \leq 3$ or T contains no T_0 for $n \geq 4$.

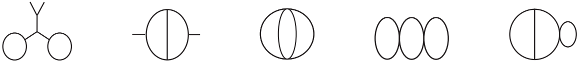
2009 Kim, Ko and P proved that $B_n\Gamma$ is a RAAG for $n \geq 5$ iff a graph Γ contains neither T_0 nor S_0.

- $B_2\Gamma$ is not a RAAG for a non-planar graph.

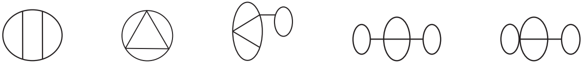


(a) T_0 (b) S_0

Braid indexes 3 and 4



(c) (d) (e) (f) (g)



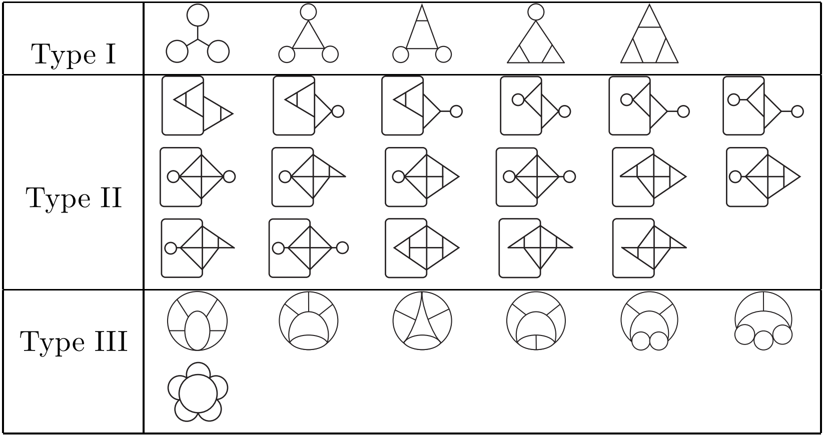
(h) (i) (j) (k) (l)

2009 Kim, La and Lee show that if Γ contains none of the above ten 3-nuclei then $B_3\Gamma$ is a RAAG.

(m) (n) (o) (p)

2011 Ko, La and P show that $B_4\Gamma$ is a RAAG iff a graph Γ contains none of the above four graphs.

Braid index 2



2011 Kim, Ko and P show that if a planar graph Γ contains one of the above graphs then $B_2\Gamma$ is not a RAAG.

On graph theory

- A graph H is *a minor* of another graph Γ if a graph isomorphic to H can be obtained from Γ by contracting some edges, deleting some edges, and deleting some isolated vertices.
- A graph H is called *a topological minor* of a graph Γ if a subdivision of H is isomorphic to a subgraph of Γ.

- **Graph minor theorem**(Robertson-Seymour theorem) Every class of graphs that is closed under minors can be defined by a finite set of minimal graphs under minor ordering.
- Note that the number of n-nuclei for $n > 4$ under minor ordering is finite.
- The number of n-nuclei for $n > 3$ under topological minor ordering is finite.

Our objectives

Question Is the number of 2-nuclei under topological minor ordering finite?

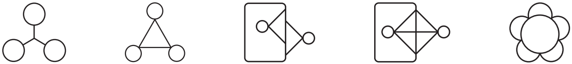
- The number of 2-nuclei is grater than sixty.
- It is not easy to describe graphs not containing one of the 2-nuclei.

Alternative question Is the property, that $B_2\Gamma$ is a RAAG, preserved under minor operators?

- If it is true then the number of 2-nuclei under minor ordering is finite.
- We conjecture that this answer is true.

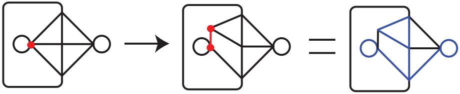
A minor closed class on $B_2\Gamma$

Lemma The set S of graphs containing none of twenty nine 2-nuclei, K_5 and $K_{3,3}$ is a minor closed class defined by the below five 2-nuclei, K_5 and $K_{3,3}$.



(q) (r) (s) (t) (u)

Sketch of the Proof: It is sufficient to check that edge extensions by graph minor theorem and the result in 2011. For example,



Corollary If $B_2\Gamma$ is a RAAG then $\Gamma \in S$.

Ingredients

The unordered discrete configuration space of a graph Γ:

$$UD_2\Gamma = \{\{\sigma_i, \sigma_j\} \subset \Gamma \mid \partial\sigma_i \cap \partial\sigma_j = \emptyset \text{ if } i \neq j\}$$

Properties

- For a sufficiently subdivided graph Γ $B_2\Gamma = \pi_1(UD_2\Gamma)$
- $UD_2\Gamma$ is a 2-dimensional non-positively cube complex and so is a $K(B_2\Gamma, 1)$ space.
- An embedding $\Gamma \hookrightarrow \Gamma'$ induces an embedding $UD_2\Gamma \hookrightarrow UD_2\Gamma'$ that is π_1-injective.

(Charney and Davis 94) If G is a RAAG, then the space K(G,1) is the full subcomplex of m-torus $\prod^m S^1$.

Corollary If $B_2\Gamma$ is a RAAG then $UD_2\Gamma$ is homotopy equivalent to a 2-dim'l full subcomplex of a m-torus.

Assume that $B_2\Gamma'$ is a RAAG. Then there is a 2-dim'l full subcomplex T of a m-torus whose π_1 is $B_2\Gamma'$ which is homotopy equivalent to $UD_2\Gamma'$.

Let Γ be a subgraph of Γ'. Then there exists an embedding $i : UD_2\Gamma \to UD_2\Gamma'$ induced by the inclusion $: \Gamma \to \Gamma'$

$$\begin{array}{ccc} UD_2\Gamma & \xrightarrow{\quad i \quad} & UD_2\Gamma' \\ {\scriptstyle h|_{UD_2\Gamma}} \downarrow & & \downarrow {\scriptstyle h} \\ h(UD_2\Gamma) & \xrightarrow{\quad \tilde{i} \quad} & T \end{array}$$

- $\pi_1(h(UD_2\Gamma))$ is a RAAG.
- $h|_{UD_2\Gamma}$ induces π_1-injective.

An edge contraction

Let Γ be a graph obtained by contracting an edge v_1-v_2 of Γ' and let v be the contracted vertex in Γ. $\overline{UD}$ is the subcomplex of $UD_2\Gamma'$ which is homeomorphic to a cube complex obtained from $UD_2\Gamma$ by an edge extension of a vertex $\{v, x\}$ to an edge $\{v_1, x\}$-$\{v_2, x\}$.

$$\begin{array}{ccc} \overline{UD} & \xrightarrow{\;i\;} & UD_2\Gamma' \\ {\scriptstyle h|_{\overline{UD}}}\downarrow & & \downarrow{\scriptstyle h} \\ h(\overline{UD}) & \xrightarrow{\;\tilde{i}\;} & T \end{array}$$

Question

Are the retractions $h|_{UD_2\Gamma}$ and $h|_{\overline{UD}}$ of the homotopy equivalence h sufficiently 'good'?

Thank you!