THE HOMFLY POLYNOMIAL AND ADMISSIBLE VALUES

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January 11, 2012

Overview

Introduction

Overview of Sec. 1

Introduction

- The HOMFLY polynomial
- Reduced polynomials
- z-span P(L;v,z)
- v-span P(L;v,z)
- Numbers of admissible values

The HOMFLY polynomial

The HOMFLY polynomial $P(L; v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ for an oriented link L is defined by the following recursive formulas:

1. For the trivial knot U_1, $P(U_1; v, z) = 1$,
2. $v^{-1}P(L_+; v, z) - vP(L_-; v, z) = zP(L_0; v, z)$.

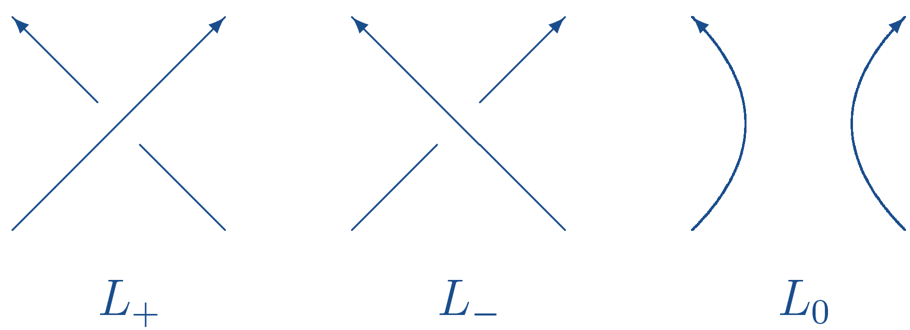

Figure 1: A skein triple

Reduced polynomials

The Jones polynomial $V(L;t)$ of L is defined by

$$V(L;t) = P(L;t, t^{1/2} - t^{-1/2}).$$

The Conway polynomial $\nabla(L;z)$ of L is defined by

$$\nabla(L;z) = P(L;1, z).$$

The Alexander polynomial $\Delta(L;t)$ of L is defined by

$$\Delta(L;t) = P(L;1, t^{1/2} - t^{-1/2}).$$

z-span P(L;v,z)

The HOMFLY polynomial $P(L;v,z)$ of L can be written as

$$P(L;v,z) = \sum_{j\in\mathbb{Z}} P_j(L;v)z^j.$$

Three values are defined as follows:

$$\max \deg_z P(L;v,z) = \max\{j; P_j(L;v) \neq 0\},$$
$$\min \deg_z P(L;v,z) = \min\{j; P_j(L;v) \neq 0\},$$
$$z\text{-span} P(L;v,z) = \max \deg_z P(L;v) - \min \deg_z P(L;v).$$

Proposition 1 (Lickorish - Millett).

$$\min \deg_z P(L;v,z) = 1 - \mu(L),$$

where $\mu(L)$ is the number of components of L.

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v-span P(L;v,z)

The HOMFLY polynomial $P(L;v,z)$ of L can be written as

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Proposition 2 (Morton, Frank - Williams).

$$b(L) \geq \frac{1}{2} v\text{-span}P(L;v,z) + 1,$$

where $b(L)$ is the braid index of L.

v-span P(L;v,z)

The HOMFLY polynomial $P(L;v,z)$ of L can be written as

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Numbers of admissible values

The table shows the numbers of admissible polynomials for the case of the knot.

v-span \ z-span	0	2	4	6	8
0	1	0	0	0	0
2	0	1	1	1	1
4	0	2	10	48+	
6	0	3			
8	0				

The structure of the polynomial

Overview of Sec. 2

The structure of the polynomial

- A characterization
- The core polynomial
- The Jones polynomial
- $z\text{-span}P = 0$
- $v\text{-span}P = 0$
- A known fact
- $v\text{-span}P = 2$
- The Conway polynomial
- $v\text{-span}P = 4$
- The Conway polynomial
- An example
- $v\text{-span}P = 6$
- $v\text{-span}P \geq 8$

A characterization

Let $U_\mu, \mu \geq 1$, be the μ-component trivial link. Put

$$\lambda_1(v,z) = P(3_1; v,z) - P(U_1; v,z) = -(1-v^2)^2 + v^2 z^2,$$
$$\lambda_\mu(v,z) = (vz)^{1-\mu}\lambda_1(v,z).$$

Theorem 3.
For a μ-comonent link L,

$$P(L; v,z) \equiv P(U_\mu; v,z) \pmod{\lambda_\mu(v,z)}.$$

Furthermore,

$$B(L; v,z) = \frac{P(L; v,z) - P(U_\mu; v,z)}{\lambda_\mu(v,z)} \in \mathbb{Z}[v^{\pm 2}, z^2]$$

is an invariant for L.

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is an invariant for L.

The core polynomial

We call $B(L; v, z)$ the core polynomial of L.

Theorem 4.
The core polynomial $B(L; v, z) \in \mathbb{Z}[v^{\pm 2}, z^2]$ for L can be recursively calcurated by the following properties:

1. *For the trivial knot U_1, $B(U_1; v, z) = 0$,*
2. *For a skein triple (L_+, L_-, L_0),*
   - (a) $B(L_+; v, z) - v^2 B(L_-; v, z) = B(L_0; v, z)$ *if* $\mu(L_+) < \mu(L_0)$,
   - (b) $B(L_+; v, z) - v^2 B(L_-; v, z) = v^2 z^2 B(L_0; v, z) + (1 - v^2)^{\mu(L_+)-2}$ *if* $\mu(L_+) > \mu(L_0)$.

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The Jones polynomial

By Theorem 3, $P(K; v, z)$ of a knot K has the following form.

$$P(K; v, z) = P(U_1; v, z) + \lambda_1(v, z)B(K; v, z).$$

Corollary 5 (Jones).
For a knot K,

$$V(K; t) - 1 = (1 - t)(1 - t^3)W(K; t),$$

where $W(K; t) \in \mathbb{Z}[t^{\pm 1}]$.

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$z\text{-span}P = 0$

By Theorem 3 and Proposition 1, we easily obtain the following.

Corollary 6.
Let L be a μ-component link.
If $z\text{-span}P(L; v, z) = 0$, then

$$P(L; v, z) = P(U_\mu; v, z).$$

z-span$P = 0$

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Corollary 6.
Let L be a μ-component link.
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$v\text{-span}P = 0$

The core polynomial $B(K; v, z)$ for a knot K can be written as

$$B(K; v, z) = v^{2a} \sum_{j \geq 0} f_{2j}(z) v^{2j}, \ a \in \mathbb{Z}, f_{2j}(z) \in \mathbb{Z}[z^2].$$

Combining the above fact with Theorem 3, we have the following.

Theorem 7.
Let K be a knot.
If $v\text{-span}P(K; v, z) = 0$, then

$$P(K; v, z) = 1 = P(U_1; v, z).$$

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Theorem 7.
Let K be a knot.
If $v\text{-span}P(K; v, z) = 0$, then

$$P(K; v, z) = 1 = P(U_1; v, z).$$

A known fact

Theorem 8 (Murakami)**.**
Let L be a link with a diagram D containing n Seifert circles. Then we have

$$a_0^{-w(D)} P(L; a_0, z) = \sum_{i=1}^{n} a_i^{-w(D)} P(L; a_i, z) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(a_0 a_j^{-1} - a_0^{-1} a_j)}{(a_i a_j^{-1} - a_i^{-1} a_j)},$$

where $a_0, a_1, \cdots, a_{n-1}$ and a_n are independent variables.

$v\text{-span}P = 2$

$T_{2,n}$ denotes the torus knot of type $(2, n)$.
If $n > 0$, then $T_{2,n}$ has n positive crossings.

Theorem 9 (Elrifai).
Let K be a knot with $v\text{-span}P(K; v, z) = 2$.
Suppose that $\min \deg_v P(K; v, z) \geq -1$.
Then, $P(K; v, z) = P(T_{2,2r+1}; v, z)$ for some $r \geq 1$.

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The Conway polynomial

Corollary 10.
Let K be a knot with $v\text{-span}P(K; v, z) = 2$.
Then,

1. $|\nabla(K; i\sqrt{2})| = 1$,
2. $|\nabla(K; i)| \leq 1$,

where i means the imaginary unit.

Remark 11. $\nabla(K; i\sqrt{2}) = \Delta(K; i), \nabla(K; i) = \Delta(K; e^{i\pi/3})$.

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Theorem 12.
Let K be a knot with $v\text{-span}P(K; v, z) = 4$.
Suppose that $\min\deg_v P(K; v, z) \geq -2$.
Then,

$$\frac{P(K; v, z) - P(T_{2,2r+1}; v, z)}{v^{2r}\lambda_1(v, z)} = \frac{P(K; v_0, z) - P(T_{2,2r+1}; v_0, z)}{v_0^{2r}\lambda_1(v_0, z)} = h(K; z),$$

where $r \geq -1$, v and v_0 are independent variables.

Remark 13. *Both sides of the above equality are the same form except the first variable. They depend on the variable z only.*

$v\text{-span}P = 4$

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Let K be a knot with $v\text{-span}P(K; v, z) = 4$.
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Let K be a knot with $v\text{-span}P(K;v,z) = 4$.
Suppose that $\min\deg_v P(K;v,z) \geq -2$.
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where $r \geq -1$, v and v_0 are independent variables.

Remark 13. *Both sides of the above equality are the same form except the first variable. They depend on the variable z only.*

Corollary 14.

Let K be a knot with $v\text{-span}P(K;v,z) = 4$.
Suppose that $\min\deg_v P(K;v,z) \geq -2$.
Then, $P(K;v,z)$ can be expressed as the following:

1. $$P(K;v,z) = P(T_{2,2r+1};v,z) + v^{2r}z^{-2}\lambda_1(v,z)(\nabla(K;z) - \nabla(T_{2,2r+1};z)),$$

2. $$P(K;v,t^{1/2}-t^{-1/2}) = P(T_{2,2r+1};v,t^{1/2}-t^{-1/2}) + v^{2r}\lambda_1(v,t^{1/2}-t^{-1/2})\frac{V(K;t) - V(T_{2,2r+1};t)}{t^{2r}(V(3_1;t)-1)}.$$

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Let K be a knot with $v\text{-span}P(K;v,z) = 4$.
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$$P(K;v,z) = P(T_{2,2r+1};v,z) + v^{2r}z^{-2}\lambda_1(v,z)(\nabla(K;z) - \nabla(T_{2,2r+1};z)),$$

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$$P(K;v,t^{1/2}-t^{-1/2}) = P(T_{2,2r+1};v,t^{1/2}-t^{-1/2}) + v^{2r}\lambda_1(v,t^{1/2}-t^{-1/2})\frac{V(K;t) - V(T_{2,2r+1};t)}{t^{2r}(V(3_1;t)-1)}.$$

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Then, $P(K;v,z)$ can be expressed as the following:

1. $$P(K;v,z)=P(T_{2,2r+1};v,z)+v^{2r}z^{-2}\lambda_1(v,z)(\nabla(K;z)-\nabla(T_{2,2r+1};z)),$$
2. $$P(K;v,t^{1/2}-t^{-1/2})=P(T_{2,2r+1};v,t^{1/2}-t^{-1/2})+v^{2r}\lambda_1(v,t^{1/2}-t^{-1/2})\frac{V(K;t)-V(T_{2,2r+1};t)}{t^{2r}(V(3_1;t)-1)}.$$

The Conway polynomial

Proposition 15.
Let K be a knot with $v\text{-span}P(K;v,z) = 4$.
Then,

1. $|\nabla(K;i\sqrt{2})| \leq 3$,
2. $|\nabla(K;i)| \leq 2$.

Corollary 16 (cf. Jones)**.**
Let K be a knot and $b(K)$ the braid index of K.
If $b(K) = 3$, then

$$|\nabla(K;i\sqrt{2})| = |\Delta(K;i)| \leq 3 \text{ and } |\nabla(K;i)| \leq 2.$$

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Let K be a knot and $b(K)$ the braid index of K.
If $b(K) = 3$, then

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An example

Example 17. *Let $K = 10_{108}$. Then,*

$$P(K; v, z) = 1 + (-2v^{-2} + 2 + 2v^2 - 2v^4)z^2 + (-v^{-2} + 3 + 3v^2 - v^4)z^4 + (1 + v^2)z^6,$$

and thus,

$$\nabla(K; z) = 1 + 4z^4 + 2z^6.$$

Since $\nabla(K; i\sqrt{2}) = 1$, it does not detect $b(K)$.
Since $\nabla(K; i) = 3$, Corollary 16 shows $b(K) \geq 4$.
This inequality is also obtained from Proposition 2 because of $v\text{-span}P(K; v, z) = 6$.
In fact, $b(K)$ is eqaul to 4.

Theorem 18.
Let K be a knot with $v\text{-span}P(K; v, z) = 6$.
Suppose that $\min \deg_v P(K; v, z) \geq -3$.
Then, $P(K; v, z)$ has either form of the following:

1. $$\frac{P(K; v, z) - 1}{v^2 \lambda_1(v, z)} = \frac{P(K; v_0, z) - 1}{v_0^2 \lambda_1(v_0, z)},$$

2. $$\frac{P(K; v, z) - P(T_{2,2r+1}; v, z)}{v^{2r} \lambda_1(v, z)}$$
$$= \sum_{i=0}^{1} \frac{P(K; v_i, z) - P(T_{2,2r+1}; v_i, z)}{v_i^{2r} \lambda_1(v_i, z)} \prod_{\substack{k=0 \\ k \neq i}}^{1} \frac{v^2 - v_k^2}{v_i^2 - v_k^2},$$

where $r \geq -1$ and v, v_0 and v_1 are mutually independent variables.

$v\text{-span}P = 6$

Theorem 18.
Let K be a knot with $v\text{-span}P(K; v, z) = 6$.
Suppose that $\min \deg_v P(K; v, z) \geq -3$.
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1.
$$\frac{P(K; v, z) - 1}{v^2\lambda_1(v, z)} = \frac{P(K; v_0, z) - 1}{v_0^2\lambda_1(v_0, z)},$$

2.
$$\frac{P(K; v, z) - P(T_{2,2r+1}; v, z)}{v^{2r}\lambda_1(v, z)}$$
$$= \sum_{i=0}^{1} \frac{P(K; v_i, z) - P(T_{2,2r+1}; v_i, z)}{v_i^{2r}\lambda_1(v_i, z)} \prod_{\substack{k=0 \\ k\neq i}}^{1} \frac{v^2 - v_k^2}{v_i^2 - v_k^2},$$

where $r \geq -1$ and v, v_0 and v_1 are mutually independent variables.

Corollary 19.
Let K be a knot with $v\text{-span}P(K; v, z) = 6$.
Suppose that $\min \deg_v P(K; v, z) \geq -3$.
Then, $P(K; v, t^{1/2} - t^{-1/2})$ can be expressed as

$$\frac{P(K; v, t^{1/2} - t^{-1/2}) - 1}{v^2 \lambda_1(v, t^{1/2} - t^{-1/2})} = \frac{\Delta(K; t) - 1}{t - 2 + t^{-1}}$$

or

$$\frac{P(K; v, t^{1/2} - t^{-1/2}) - P(T_{2,2r+1}; v, t^{1/2} - t^{-1/2})}{v^{2r} \lambda_1(v, t^{1/2} - t^{-1/2})}$$
$$= \frac{\Delta(K; t) - \Delta(T_{2,2r+1}; t)}{t - 2 + t^{-1}} \cdot \frac{v^2 - t^2}{1 - t^2} + \frac{V(K; t) - V(T_{2,2r+1}; t)}{t^{2r}(V(3_1; t) - 1)} \cdot \frac{v^2 - 1}{t^2 - 1}.$$

Corollaries 14 and 19 correspond to the following fact.

Theorem 20 (stoimenow). *Let β be a braid and $L = \hat{\beta}$. If $v\text{-span}P(L; v.z) \le 6$, then $\Delta(L;t)$ and $V(L;t)$ together with the exponent sum $[\beta]$ determine $P(L; v, z)$.*

v-span$P \geq 8$

Theorem 21.
Let K be a knot with $v\text{-span}P(K;v,z) = 2n, n \geq 4$.
Suppose that $\min\deg_v P(K;v,z) \geq -n$.
Then $P(K;v,z)$ has any of the following three forms:

1.
$$\frac{P(K;v,z)-1}{v^{2(n-2)}\lambda_1(v,z)} = \frac{P(K;v_0,z)-1}{v_0^{2(n-2)}\lambda_1(v_0,z)},$$

2.
$$\frac{P(K;v,z)-1}{v^{2s}\lambda_1(v,z)} = \sum_{i=0}^{N} \frac{P(K;v_i,z)-1}{v_i^{2s}\lambda_1(v_i,z)} \prod_{\substack{k=0 \\ k\neq i}}^{N} \frac{v^2 - v_k^2}{v_i^2 - v_k^2},$$

where $-n/2 \leq s \leq n-3$, $N = n-2-\max\{s,0\}$ and $v, v_0, \cdots, v_N$ are mutually independent variables, 3.

$$\frac{P(K;v,z)-P(T_{2,2r+1};v,z)}{v^{2r}\lambda_1(v,z)}$$
$$=\sum_{i=0}^{n-2}\frac{P(K;v_i,z)-P(T_{2,2r+1};v,z)}{v_i^{2r}\lambda_1(v_i,z)}\prod_{\substack{k=0\\k\neq i}}^{n-2}\frac{v^2-v_k^2}{v_i^2-v_k^2},$$

where $r \geq 1$ and $v, v_0, \cdots, v_{n-2}$ are mutually independent variables.

Admissible values

Overview of Sec. 3

Admissible values

- $v\text{-span}P = 0$
- $v\text{-span}P = 2$
- $v\text{-span}P = 4$
- Question
- $z\text{-span}P = 0$
- $z\text{-span}P = 2$

$v\text{-span}P = 0$

Theorem 22 (Theorem 7)**.**
Let K be a knot.
If $v\text{-span}P(K; v, z) = 0$, then

$$P(K; v, z) = 1 = P(U_1; v, z).$$

$v\text{-span}P = 2$

By Theorem 9, we have

Theorem 23.
Let K be a knot.
If $v\text{-span}P(K; v, z) = 2$, then

$$P(K; v, z) \in \{P(T_{2,(2r+1)\varepsilon}; v, z); r \geq 0, \varepsilon = \pm 1\}.$$

Remark 24. *Since* $\max\deg_z P(T_{2,(2r+1)\varepsilon}; v, z) = 2r$*, Proposition 1 gives* $z\text{-span}P(T_{2,(2r+1)\varepsilon}; v, z) = 2r$*.*

v-span$P = 4$

Proposition 25.
Let K be a knot with v-span$P(K; v, z) = 4$. Then,

1. *(Corollary 6)* z-span$P(K; v, z) > 0$
2. *If* z-span$P(K; v, z) = 2$, *then*

$$P(K; v, z) \in \{P(4_1; v, z), P(5_2; v, z)\},$$

3. *If* z-span$P(K; v, z) = 4$, *then*

$$P(K; v, z) \in \{P(K; v, z); K = 6_2, 6_3, 7_3, 7_5, 8_{20}, 8_{21}, 9_{42}, 9_{49}, 3_1 \sharp 3_1, 3_1 \sharp (3_1!)\}.$$

$v\text{-span}P = 4$

Proposition 25.
Let K be a knot with $v\text{-span}P(K; v, z) = 4$. Then,

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$$P(K; v, z) \in \{P(K; v, z); K = 6_2, 6_3, 7_3, 7_5, 8_{20}, 8_{21}, 9_{42}, 9_{49}, 3_1\sharp 3_1, 3_1\sharp(3_1!)\}.$$

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v-span$P = 4$ (Cont'd)

Proposition 26.
Let K be a knot with v-span$P(K; v, z) = 4$.
If z-span$P(K; v, z) = 6$, then $P(K; v, z)$ can be obtained from one of at most 24 types of possible polynomials in three variables v, z and c by replacing c with some integer.

Example 27.

$$R(v,z,c) = (8-2c)v^6 + (-11+4c)v^8 + (4-2c)v^{10} + \{(14-3c)v^6 + (-16+8c)v^8 + (4-3c)v^{10}\}z^2 + \{(7-c)v^6 + (-7+5c)v^8 + (1-c)v^{10}\}z^4 + \{v^6 + (-1+c)v^8\}z^6.$$

$R(v,z,1) = P(3_1\sharp 5_1; v, z)$,
$R(v,z,2) = P(9_{16}; v, z)$.

Proposition 26.
Let K be a knot with $v\text{-span}P(K;v,z) = 4$.
If $z\text{-span}P(K;v,z) = 6$, then $P(K;v,z)$ can be obtained from one of at most 24 types of possible polynomials in three variables v, z and c by replacing c with some integer.

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$$\begin{aligned} R(v,z,c) = {} & (8-2c)v^6 + (-11+4c)v^8 + (4-2c)v^{10} \\ & + \{(14-3c)v^6 + (-16+8c)v^8 + (4-3c)v^{10}\}z^2 \\ & + \{(7-c)v^6 + (-7+5c)v^8 + (1-c)v^{10}\}z^4 \\ & + \{v^6 + (-1+c)v^8\}z^6. \end{aligned}$$

$$R(v,z,1) = P(3_1\sharp 5_1; v,z),$$
$$R(v,z,2) = P(9_{16}; v,z).$$

Question

As I mentioned in Section 1, The nimber of admissible polynomials with $v\text{-span}P = 4$ and $z\text{-span}P = 6$ is not decided yet.

Question 28.

$$\sharp\{P(K; v, z); v\text{-span}P = 4, z\text{-span}P = 6\} < \infty?$$

$z\text{-span}P = 0$

Proposition 29 (Corollary 6)**.**
Let K be a knot.
If $z\text{-span}P(K; v, z) = 0$, then $P(K; v, z) = P(U_1; v, z)$.

z-span$P = 2$

Proposition 30.
Let K be a knot. If $z\text{-span}P(K;v,z) = 2$, *then*

$$P(K;t,z) = 1 - \frac{(1+t)^2(1-V(K;t))}{t^2+t+1} + \frac{t^2(1-V(K;t))}{(t-1)(t^3-1)}z^2.$$

Proposition 31.
Let K be a knot with $z\text{-span}P(K;v,z) = 2$.
If $v\text{-span}P(K;v,z) = 6$, *then*

$$P(K;v,z) \in \{P(K;v,z); K = 6_1, 7_2, 9_{46}\}.$$

z-span$P = 2$

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Polynomials of some Knots

Overview of Sec. 4

Polynomials of some Knots

- Amphicheiral knot
- Alternating and amphicheiral
- The Alexander polynomial

Amphicheiral knot

Proposition 32.
Let K be an amphicheiral knot. If $b(K) \leq 4$, then

$$V(K;t) = 1 - (t^{-1} + 1 + t)(\Delta(K;t) - 1).$$

Example 33. *Let $K = 8_3$. Then,*

$$V(K;t) = t^{-4} - t^{-3} + 2t^{-2} - 3t^{-1} + 3 - 3t + 2t^2 - t^3 + t^4,$$
$$\Delta(K;t) = -4t^{-1} + 9 - 4t,$$

and thus,

$$V(K;t) \neq 1 - (t^{-1} + 1 + t)(\Delta(K;t) - 1).$$

Hence, $b(K) \geq 5$. In fact, $b(K)$ is eqaul to 5.

Amphicheiral knot

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Hence, $b(K) \geq 5$. In fact, $b(K)$ is eqaul to 5.

Alternating and amphicheiral

Corollary 34.
Let K be a non-trivial alternating, amphicheiral knot.
If $b(K) \leq 4$, then $cr(K) = 2(g(K) + 1)$.

Example 35.
Let $K = 8_3$. K is alternating and ampiceiral.
Since $cr(K) = 8$ and $g(K) = 1$, $b(K) \geq 5$.

K	8_{12}	10_{33}	10_{37}	10_{43}	10_{45}	10_{81}	10_{88}	10_{115}
$g(K)$	2	2	2	3	3	3	3	3
$b(K)$	5	5	5	5	5	5	5	5

Alternating and amphicheiral

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Let K be a non-trivial alternating, amphicheiral knot.
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K	8_{12}	10_{33}	10_{37}	10_{43}	10_{45}	10_{81}	10_{88}	10_{115}
$g(K)$	2	2	2	3	3	3	3	3
$b(K)$	5	5	5	5	5	5	5	5

The Alexander polynomial

Proposition 36.
Let K be a knot with $b(K) \leq 4$.
Then, $\Delta(K;t)$ has any of the following four forms:

1.
$$\Delta(K;t) = \Delta(T_{2,2r+1};t),$$

2.
$$\frac{\Delta(K;t)-1}{\Delta(3_1;t)-1} = t^{-2}\frac{V(K;t)-1}{V(3_1;t)-1},$$

3.
$$\frac{\Delta(K;t)-\Delta(T_{2,2r+1};t)}{\Delta(3_1;t)-1} = t^{-2r}\frac{V(K;t)-V(T_{2,2r+1};t)}{V(3_1;t)-1},$$

4.

$$\frac{\Delta(K;t)-\Delta(T_{2,2r+1};t)}{\Delta(3_1;t)-1} = \frac{1}{t+t^{-1}}\sum_{\varepsilon\in\{\pm1\}} t^{-(2r+1)\varepsilon}\frac{V(K;t^{\varepsilon})-V(T_{2,2r+1};t^{\varepsilon})}{V(3_1;t^{\varepsilon})-1}.$$

The last formula corresponds to Murakami's observation.

Proposition 37 (Murakami).
Let L be a link with a diagram D. If D has 4 Seifert circles, then $\Delta(L;t)$ can be expressed in terms of $V(L;t)$ and $V(L;t^{-1})$.