Arc index of pretzel knots of type $(-p, q, r)$

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Arc presentation and arc index

An *arc presentation* of a knot or a link L is an ambient isotopic image of L contained in the union of finitely many half planes, called *pages*, with a common boundary line in such a way that each half plane contains a properly embedded single arc.

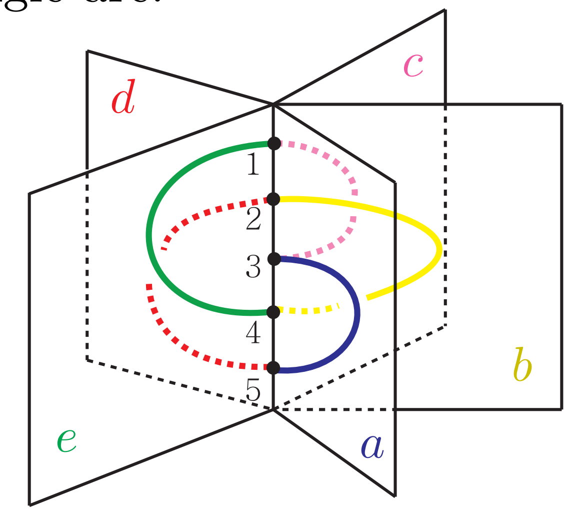


Figure 1: An arc presentation of the trefoil knot

The minimal number of pages among all arc presentations of a link L is called the *arc index* of L and is denoted by $\alpha(L)$.

Motivation

- [Cromwell,1995] Every link admits an arc presentation.

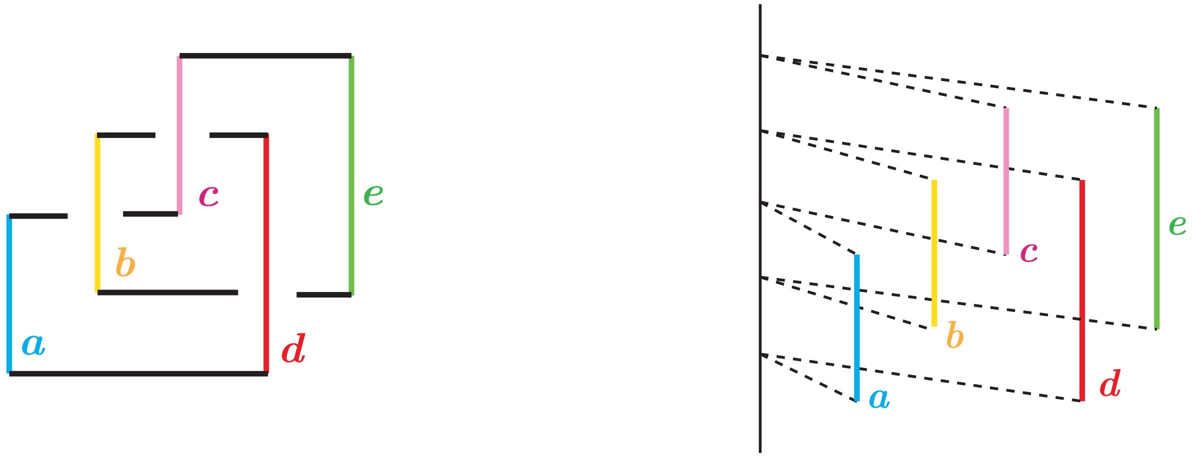


Figure 2: Cromwell diagram

Motivation : Arc index and crossing number

- [Thistlethwaite, 1988]
  If L is an alternating link, then $\text{spread}_a(F_L(a,z)) \geq c(L)$.

- [Morton-Beltrami, 1998]
  For any link L, $\alpha(L) \geq \text{spread}_a(F_L(a,z)) + 2$.

- [Bae-Park, 2000]
  If L is a non-split alternating link, then $\alpha(L) \leq c(L) + 2$.

- [Jin-Park, 2010]
  A prime link L is nonalternating *if and only if* $\alpha(L) \leq c(L)$.

Pretzel knots of type $(-p, q, r)$

Pretzel knot of type $(-p, q, r)$, denoted by $P(-p, q, r)$, is a knot that admits a diagram D composed of three integer tangles with $-p, q$ and r half-twists as in Figure 3.

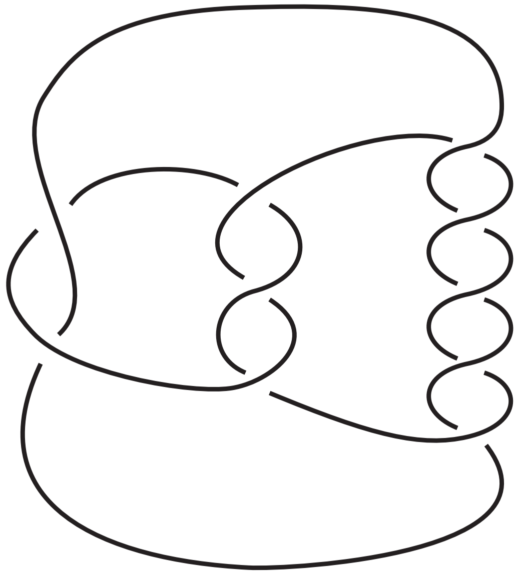


Figure 3: $P(-2, 3, 5)$

Properties of $P(-p, q, r)$

- $P(p_1, p_2, p_3)$ is a knot if and only if at most one of p_1, p_2, p_3 is an even number.

- For any integers p_1, p_2, p_3, two links $P(p_1, p_2, p_3)$ and $P(p_i, p_j, p_k)$ are equivalent whenever $\{i, j, k\} = \{1, 2, 3\}$.

So we can assume without loss of generality that $r \geq q$ in $P(-p, q, r)$.

Main results : Arc index of $P(-p, q, r)$

- [**Beltrami-Cromwell**] If $K = P(-p, q, r)$ is a knot with $p, q, r \geq 2$, then

$$\alpha(K) \leq c(K) = p + q + r.$$

Theorem 1 *If $K = P(-2, q, r)$ is a knot with $q, r \geq 3$, then*

$$\alpha(K) \leq c(K) - 1 = q + r + 1.$$

Theorem 2 *If $K = P(-p, 2, r)$ is a knot with $p, r \geq 3$, then*

$$\alpha(K) = c(K) = p + r + 2.$$

Theorem 3 *If $K = P(-p, 3, r)$ is a knot with $p, r \geq 3$, then*

$$\alpha(K) = c(K) - 1 = p + r + 2.$$

Theorem 4 *If $K = P(-p, q, r)$ is a knot with $p \geq 3$ and $q, r \geq 4$, then*

$$\alpha(K) \leq c(K) - 2 = p + q + r - 2.$$

Theorem 5 *If $K = P(-p, 4, r)$ is a knot with $p, r \geq 5$, then*

$$\alpha(K) = c(K) - 2 = p + r + 2.$$

How to determine the arc index of $K = P(-p, q, r)$?

Our strategy is ...

- For the lower bound of $\alpha(K)$, we compute the

$$\text{spread}_a(F_K(a, z)).$$

  ([Morton-Beltrami] For any link L, $\alpha(L) \geq \text{spread}_a(F_L(a, z)) + 2$.)

- For the upper bound of $\alpha(K)$,
  we show an arc presentation of K which is presented with minimal arcs.

Kauffman polynomial $F_L(a, z)$

The *Kauffman polynomial* $F_L(a, z)$ of an oriented knot or link L is defined by

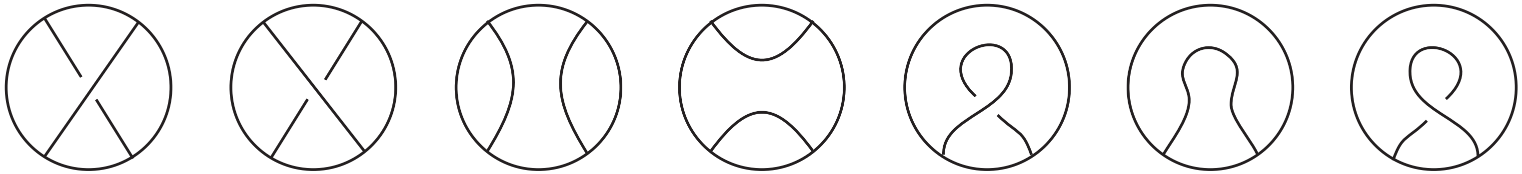
$$F_L(a, z) = a^{-w(D)} \Lambda_D(a, z)$$

where D is a diagram of L, $w(D)$ the writhe of D and $\Lambda_D(a, z)$ the polynomial determined by the rules (K1), (K2) and (K3).

(K1) $\Lambda_O(a, z) = 1$ where O is the trivial knot diagram.

(K2) $\Lambda_{D_+}(a, z) + \Lambda_{D_-}(a, z) = z(\Lambda_{D_0}(a, z) + \Lambda_{D_\infty}(a, z))$.

(K3) $a\, \Lambda_{D_\oplus}(a, z) = \Lambda_D(a, z) = a^{-1} \Lambda_{D_\ominus}(a, z)$.



D_+ D_- D_0 D_∞ $D_\oplus$ D $D_\ominus$

Properties of $\Lambda_D(a, z)$

(K4) If D is a connected sum of D_1 and D_2, then

$$\Lambda_D(a, z) = \Lambda_{D_1}(a, z)\, \Lambda_{D_2}(a, z).$$

(K5) If D is a split union of D_1 and D_2, then

$$\Lambda_D(a, z) = (z^{-1}a - 1 + z^{-1}a^{-1})\, \Lambda_{D_1}(a, z)\, \Lambda_{D_2}(a, z).$$

$\text{spread}_a(F_L)$

The Laurent degree in the variable a of the Kauffman polynomial $F_L(a, z) = a^{-w(D)}\Lambda_D(a, z)$ is denoted by $\text{spread}_a(F_L)$ and defined by the formula

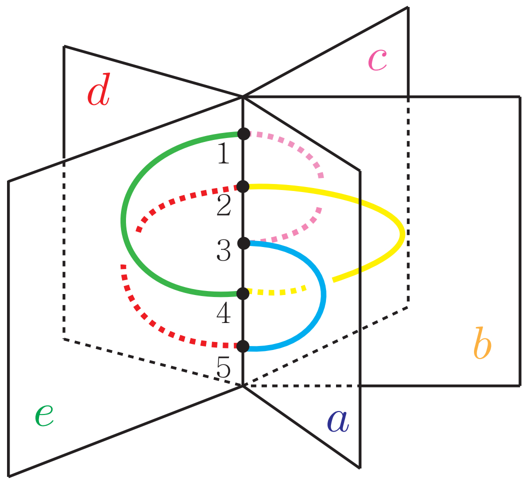
$$\text{spread}_a(F_L) = \text{max-deg}_a(F_L) - \text{min-deg}_a(F_L).$$

Notice that $\text{spread}_a(F_L) = \text{spread}_a(\Lambda_D)$ for any diagram D of L.

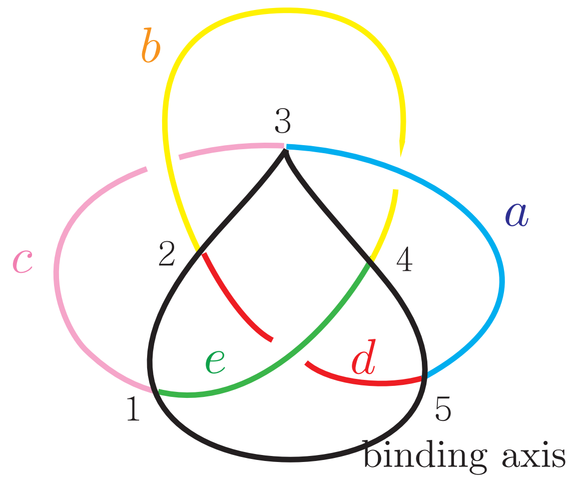
- [**Morton-Beltrami**] Let L be a link. Then

$$\alpha(L) \geq \text{spread}_a(F_L) + 2.$$

An representation of arc presentation



(a)



(b)

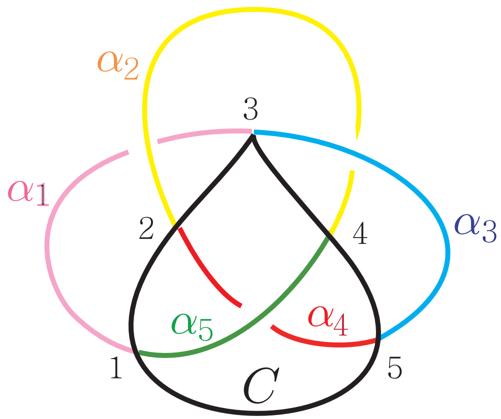
Figure 4: An representation of arc presentation

Arc presentation

Let D be a diagram of a knot or a link L. Suppose that there is a simple closed curve C meeting D in k distinct points which divide D into k arcs $\alpha_1, \alpha_2, \ldots, \alpha_k$ with the following properties:

1. Each α_i has no self-crossing.
2. If α_i crosses over α_j at a crossing, then $i > j$ and it crosses over α_j at any other crossings with α_j.
3. For each i, there exists an embedded disk d_i such that $\partial d_i = C$ and $\alpha_i \subset d_i$.
4. $d_i \cap d_j = C$, for distinct i and j.

Then the pair (D, C) is called an *arc presentation* of L with k arcs.



Theorem 1 : If $K = P(-2, q, r)$ is a knot with $q, r \geq 3$, then

$$\alpha(K) \leq c(K) - 1 = q + r + 1.$$

(Sketch of proof)

Figure 5 shows an arc presentation of $K = P(-2, q, r)$ with $q + r + 1$ arcs. Therefore $\alpha(K) \leq c(K) - 1 = q + r + 1$.

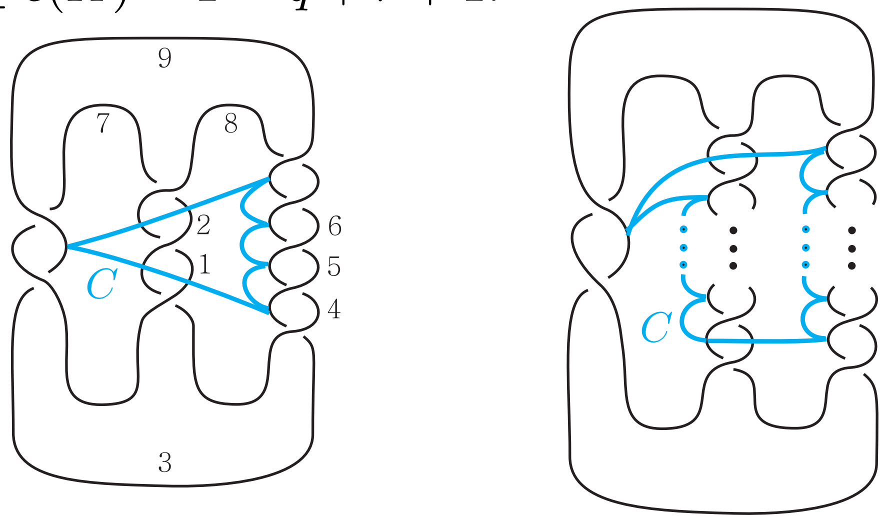


Figure 5: Arc presentations of $P(-2, 3, 5)$ and $P(-2, q, r)$

Theorem 2 : If $K = P(-p, 2, r)$ is a knot with $p, r \geq 3$, then

$$\alpha(K) = c(K) = p + r + 2.$$

(Sketch of proof)

From (K1) to (K5), we can compute $\Lambda_{\mathbf{P}(-p,2,r)}$ as follows.

$$\Lambda_{\mathbf{P}(-p,2,r)} = -\Lambda_{\mathbf{P}(-p,2,r-2)} + z\Lambda_{\mathbf{P}(-p,2,r-1)} + za^{-(r-1)}\Lambda_{\mathbf{P}(-p,2)}$$

$$\cdots$$

$$= (z^{r+2} + t_{r+1}z^{r+1} + \cdots)a^p + \cdots + (z^{p-2} + s_{p-3}z^{p-3} + \cdots)a^{-r}$$

Therefore, $\alpha(K) \geq \text{spread}_a(F_K) + 2 = p + r + 2.$

Figure 6 shows an arc presentation of $K = P(-p, 2, r)$ with $p + r + 2$ arcs. Therefore $\alpha(K) \leq c(K) = p + r + 2$.

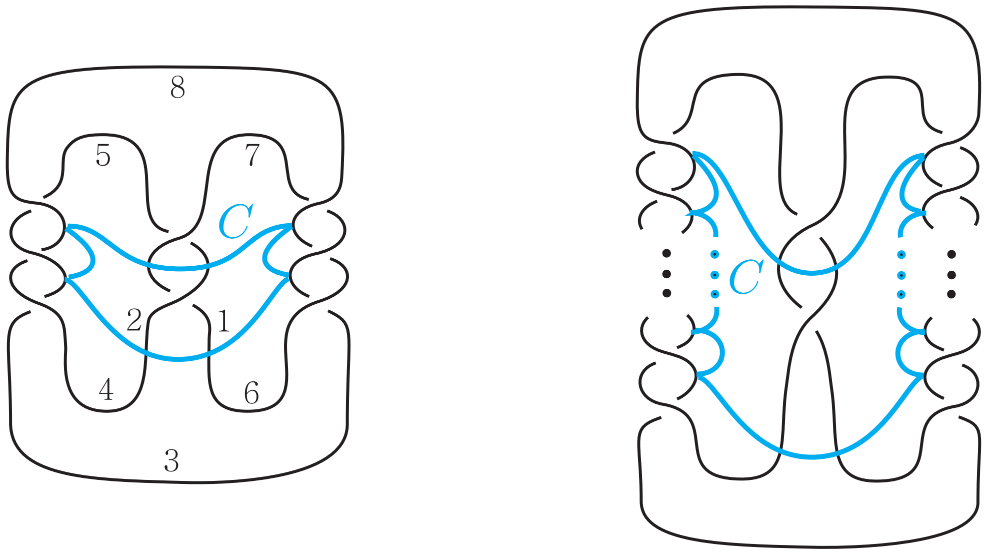


Figure 6: Arc presentations of $P(-3, 2, 3)$ and $P(-p, 2, r)$

So we have $\alpha(K) = c(K) = p + r + 2$.

Theorem 3 : If $K = P(-p, 3, r)$ is a knot with $p, r \geq 3$, then

$$\alpha(K) = c(K) - 1 = p + r + 2.$$

(Sketch of proof)

From (K1) to (K5), we can compute $\Lambda_{\mathbf{P}(-p,3,r)}$ as follows.

$$\begin{aligned}\Lambda_{\mathbf{P}(-p,3,r)} &= -\Lambda_{\mathbf{P}(-p,3,r-2)} + z\Lambda_{\mathbf{P}(-p,3,r-1)} + za^{-(r-1)}\Lambda_{\mathbf{P}(-p,3)} \\ &\cdots \\ &= (z^{r+3} + t_{r+2}z^{r+2} + \cdots)a^{p} + \cdots + (z^{p-3} + s_{p-4}z^{p-4} + \cdots)a^{-r}\end{aligned}$$

Therefore, $\alpha(K) \geq \mathrm{spread}_a(F_K) + 2 = p + r + 2.$

Figure 7 shows an arc presentation of $P(-p, 3, r)$ with $p + r + 2$ arcs. Therefore $\alpha(K) \leq c(K) - 1 = p + r + 2$.

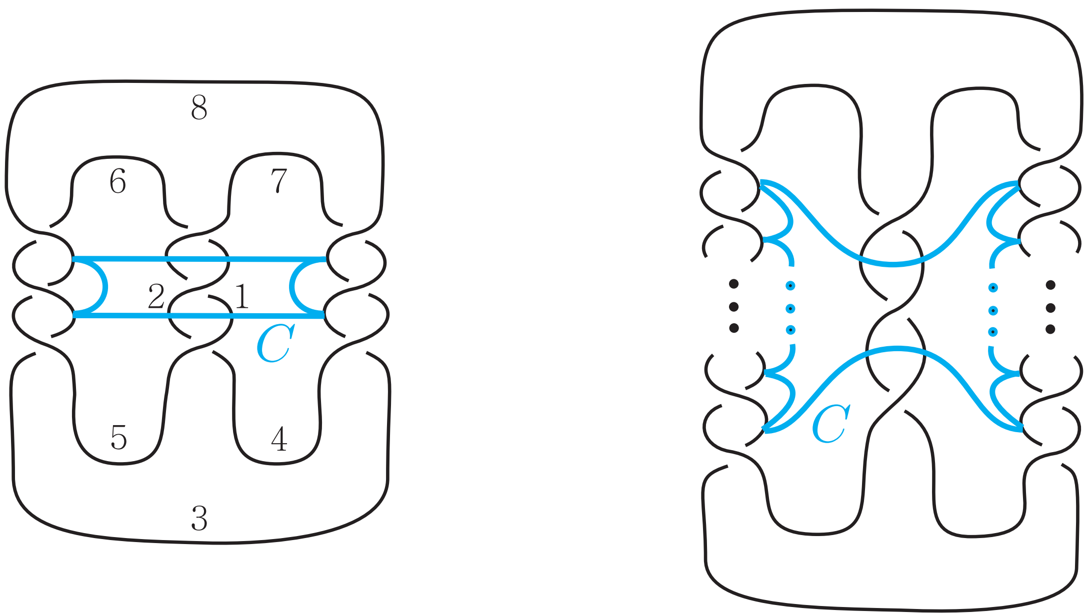


Figure 7: Arc presentations of $P(-3, 3, 3)$ and $P(-p, 3, r)$

So we have $\alpha(K) = c(K) - 1 = p + r + 2$.

Theorem 4 : If $K = P(-p, q, r)$ is a knot with $p \geq 3$ and $q, r \geq 4$, then $\alpha(K) \leq c(K) - 2 = p + q + r - 2$.

(Sketch of proof)
Figure 8 shows an arc presentation of $K = P(-p, q, r)$ with $p + q + r - 1$ arcs after Reidemeister moves of type three twice. However two arcs meeting at the vertex v of the arc presentation can be exchanged only one edge in a new disk d such that $\partial d = C$. Hence $\alpha(K) \leq c(K) - 2 = p + q + r - 2$.

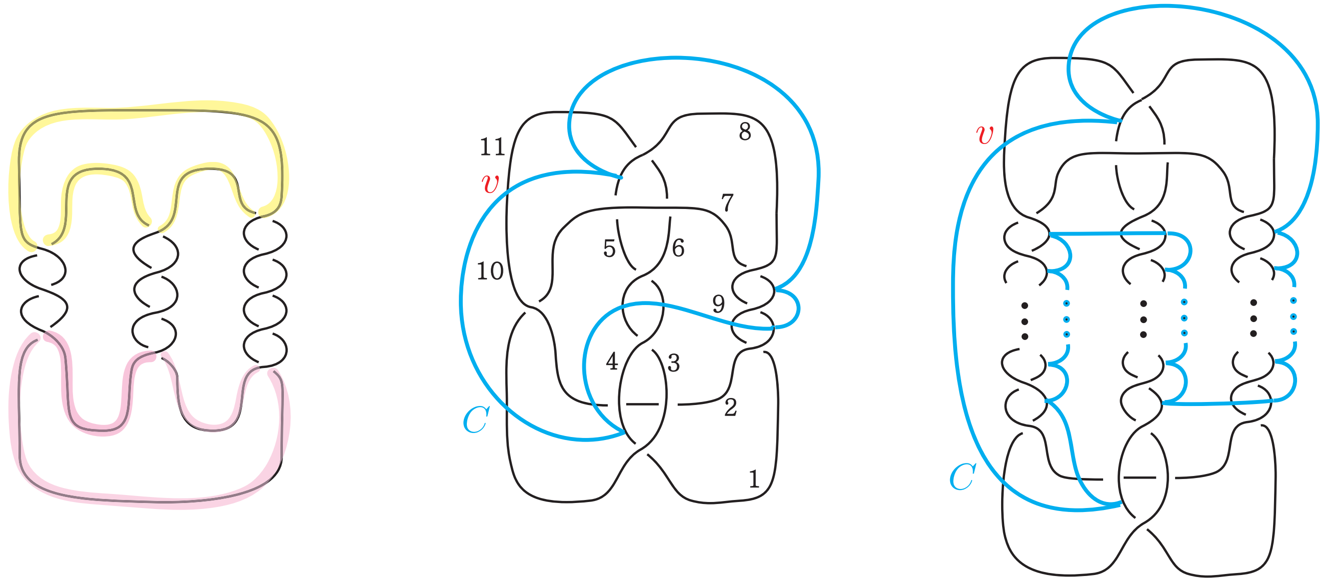


Figure 8: Arc presentations of $P(-3, 4, 5)$ and $P(-p, q, r)$

Theorem 5 : If $K = P(-p, 4, r)$ is a knot with $p, r \geq 5$, then

$$\alpha(K) = c(K) - 2 = p + r + 2.$$

(Sketch of proof)

From (K1) to (K5), we can compute $\Lambda_{\mathbf{P}(-p,q,r)}$ as follows.

$$\Lambda_{\mathbf{P}(-p,4,r)} = -\Lambda_{\mathbf{P}(-p,4,r-2)} + z\Lambda_{\mathbf{P}(-p,4,r-1)} + za^{-(r-1)}\Lambda_{\mathbf{P}(-p,4)}$$

$$\cdots$$

$$= (z^{r+4} + t_{r+3}z^{r+3} + \cdots)a^{p} + \cdots + (z^{p-4} + s_{p-5}z^{p-5} + \cdots)a^{-r}$$

Therefore, $\alpha(K) \geq \text{spread}_a(F_K) + 2 = p + r + 2$. Since $K = P(-p, 4, r)$ has an arc presentation with $p + r + 2$ arcs by Theorem 4,
$\alpha(K) = c(K) - 2 = p + r + 2.$

Thank you very much!