Geometric automorphisms of braid groups on surfaces

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Configuration space and braid group I

Let Σ be a connected, orientable surface with $\partial\Sigma = \emptyset$, $\chi(\Sigma) < 0$.

- Ordered configuration space $F_n(\Sigma)$

$$F_n(\Sigma) = \{(z_1, \ldots, z_n) \in \Sigma^n | z_i \neq z_j \text{ if } i \neq j\}.$$

- (Unordered) configuration space $B_n(\Sigma)$

$$B_n(\Sigma) = \{\{z_1, \ldots, z_n\} \subset \Sigma | z_i \neq z_j \text{ if } i \neq j\} = F_n(\Sigma)/\mathbf{S}_n.$$

Then $p : F_n(\Sigma) \to B_n(\Sigma)$ is a covering map with deck transformation group $\mathbf{S}_n$, the symmetric group on n letters.

Configuration space and braid group II

Let $\mathbf{z}^0 = (z_1^0, \ldots, z_n^0)$, $\bar{\mathbf{z}}^0 = \{z_1^0, \ldots, z_n^0\}$ be basepoints.

- *pure n-braid group* $\mathbf{P}_n(M, \mathbf{z}^0) = \pi_1(F_n(M), \mathbf{z}^0)$.
- *(full) n-braid group* $\mathbf{B}_n(M, \bar{\mathbf{z}}^0) = \pi_1(B_n(M), \bar{\mathbf{z}}^0)$.

Then there is a short exact sequence as follows.

$$1 \longrightarrow \mathbf{P}_n(\Sigma, \mathbf{z}^0) \xrightarrow{p_*} \mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0) \xrightarrow{\rho_n} \mathbf{S}(\bar{\mathbf{z}}^0) \longrightarrow 1 \qquad \text{(BS)}$$

where $\mathbf{S}(\bar{\mathbf{z}}^0)$ is the symmetric group on $\bar{\mathbf{z}}^0$.
Note that ρ_n depends only on $\bar{\mathbf{z}}^0$, and so is $\mathbf{P}_n(\Sigma, \mathbf{z}^0) = \operatorname{Ker} \rho_n$.

Question

What is $\mathrm{Aut}(\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0))$?

Mapping class group and Geometric automorphism

$\mathcal{M}(\Sigma, \bar{\mathbf{z}}^0)$: *mapping class group* on $(\Sigma, \bar{\mathbf{z}}^0)$

$$\mathcal{M}(\Sigma, \bar{\mathbf{z}}^0) = \left\{ \bar{f} : (\Sigma, \bar{\mathbf{z}}^0) \to (\Sigma, \bar{\mathbf{z}}^0) | \bar{f} \text{ is homeo.} \right\} / \textit{isotopy}$$

Let $\bar{f} \in \mathcal{M}(\Sigma, \bar{\mathbf{z}}^0)$. Then $\bar{f}$ induces

$$\bar{f}_* \in \mathrm{Aut}(\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0)).$$

Definition 1

An automorphism $\phi \in \mathrm{Aut}(\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0))$ is *geometric* if there exists an embedding $\bar{f} \in \mathcal{M}(\Sigma, \bar{\mathbf{z}}^0)$ inducing ϕ.

Known facts

Theorem 2

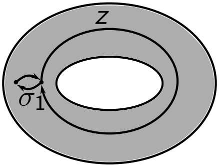
$\mathcal{M}(\Sigma, \bar{\mathbf{z}}^0)$ *embeds into* $\mathrm{Aut}(\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0))$.

Theorem 3 (Ivanov)

If Σ *is closed, then for* $n \geq 2$,

$$\mathrm{Aut}(\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0)) \simeq \mathcal{M}(\Sigma, \bar{\mathbf{z}}^0)$$

Automorphism which is NOT geometric



$$\mathbf{B}_2(A) = \langle \sigma_1, z | [z, \sigma_1 z \sigma_1] \rangle.$$

Table: 2-braid group on annulus

$\phi \in \mathrm{Aut}(\mathbf{B}_2(A))$ defined by $\phi(\sigma_1) = z, \phi(z) = \sigma_1$ is not geometric.

Question

What is the necessary and sufficient condition for $\phi \in \mathrm{Aut}(\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0))$ to be geometric?

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What is the necessary and sufficient condition for $\phi \in \mathrm{Aut}(\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0))$ to be geometric?

From now on, we assume that $\bar{f} \in \mathcal{M}(\Sigma, \bar{\mathbf{z}}^0)$, $\phi \in \mathrm{Aut}(\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0))$, and fix $\mathbf{z}^0 \in p^{-1}(\bar{\mathbf{z}}^0)$.

Necessary condition I

Theorem 4 (Ivanov)

$\mathbf{P}_n(\Sigma, \mathbf{z}^0)$ *is a characteristic subgroup of* $\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0)$ *for any* $n \neq 2$.

Therefore there exists $\rho_n(\bar{f}) \in \mathrm{Aut}(\mathbf{S}(\bar{\mathbf{z}}^0))$ that makes the following diagram commutative.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{P}_n(\Sigma, \mathbf{z}^0) & \xrightarrow{p_*} & \mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0) & \xrightarrow{\rho_n} & \mathbf{S}(\bar{\mathbf{z}}^0) & \longrightarrow & 1 \\
 & & \downarrow f_* & & \downarrow \bar{f}_* & & \downarrow \rho_n(\bar{f}) & & \\
1 & \longrightarrow & \mathbf{P}_n(\Sigma, \mathbf{z}'^0) & \xrightarrow{p_*} & \mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0) & \xrightarrow{\rho_n} & \mathbf{S}(\bar{\mathbf{z}}^0) & \longrightarrow & 1
\end{array}$$

where $\mathbf{z}'^0 = f(\mathbf{z}^0)$.

Necessary condition II

For the index set $I = \{i_1, \ldots, i_k\} \subsetneq \{1, \ldots, n\}$ with $i_1 < \cdots < i_k$, let

$$\mathbf{z}_I = (z_{i_1}, \ldots, z_{i_k}), \qquad \bar{\mathbf{z}}_I = \{z_{i_1}, \ldots, z_{i_k}\}.$$

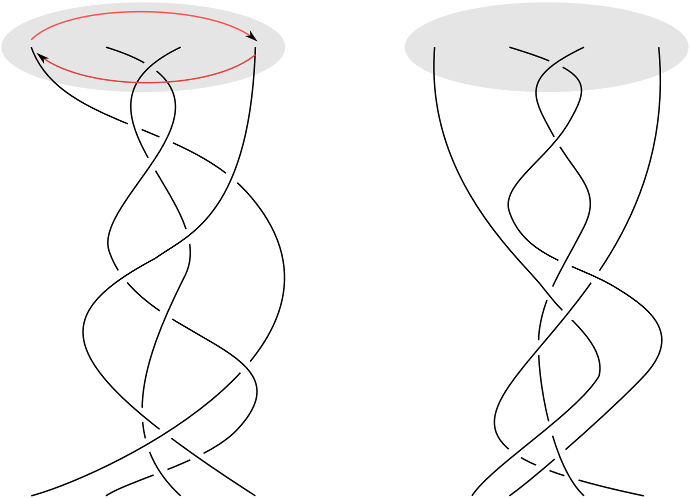
We define the *I-forgetting map* $p_I : F_n(\Sigma) \to F_{n-k}(\Sigma)$ as

$$p_I(\mathbf{z}) = \mathbf{z}_{I^c}$$

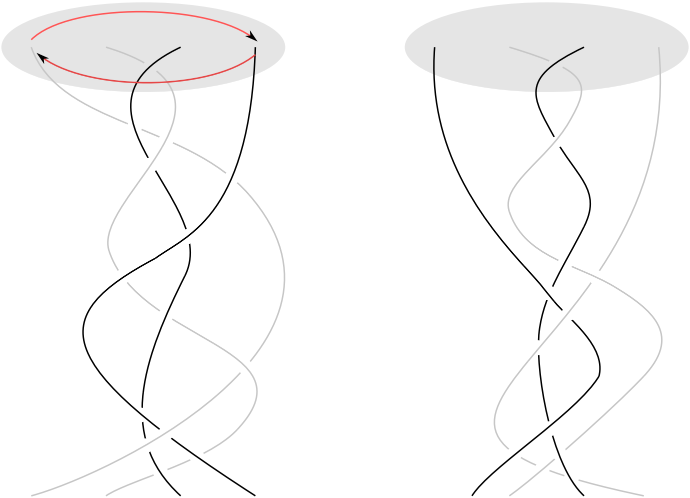
where I^c is the complement of I.
Then there exists $J \subsetneq \{1, \ldots, n\}$, we get

$$f \circ p_I = p_J \circ f.$$

Necessary condition III



Necessary condition IV



The induced maps $(p_I)_*, (p_J)_*$ make the below diagram commutative.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Ker}(p_I)_* & \longrightarrow & \mathbf{P}_n(\Sigma, \mathbf{z}^0) & \xrightarrow{(p_I)_*} & \mathbf{P}_{n-k}(\Sigma, \mathbf{z}^0_{I^c}) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle f_*|_{\mathrm{Ker}(p_I)_*}} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*\circ(p_I)_*} & & \\ 1 & \longrightarrow & \mathrm{Ker}(p_J)_* & \longrightarrow & \mathbf{P}_n(\Sigma, \mathbf{z}'^0) & \xrightarrow{(p_J)_*} & \mathbf{P}_{n-k}(\Sigma, \mathbf{z}'^0_{J^c}) & \longrightarrow & 1 \end{array}$$

Note that

$$\mathrm{Ker}(p_I)_* = \mathbf{P}_k(\Sigma \setminus \bar{z}^0_{I^c}, z^0_I),$$

$$\mathrm{Ker}(p_J)_* = \mathbf{P}_k(\Sigma \setminus \bar{z}'^0_{J^c}, z'^0_J).$$

Necessary condition VI

Necessary condition (NC)

Let $\phi \in \mathrm{Aut}(\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0))$. For any $I \subsetneq \{1, \ldots, n\}$, there exists J such that

$$\phi|_{\mathrm{Ker}(p_I)_*} = \mathrm{Ker}(p_J)_*.$$

Let $\mathrm{Aut}_{NC}(\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0))$ be a group of automorphisms satisfying (NC).

Main Theorem

For $n \geq 3$,

$$\mathcal{M}(\Sigma, \bar{\mathbf{z}}^0) \simeq Aut_{NC}(\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0))$$

Lemma 5

For any $I \subsetneq \{1, \ldots, n\}$,

$$\phi|_{Ker(p_I)_*} = Id \iff \phi \equiv Id.$$

Theorem 6 (well known)

Let Σ *be a closed surface and* $\phi \in \mathrm{Aut}(\pi_1(\Sigma \setminus \{z_1^0, \ldots, z_{n-1}^0\}, z_n^0))$, *and* ζ_i *be a loop based at* z_n^0 *surrounding* z_i^0 *once. Then* ϕ *is ori. pres. geometric if and only if* ϕ *fixes the set of conjugacy classes of* ζ_i*'s.*

By composition with suitable geometric automorphism, we may assume that for all I,

$$\phi \circ (p_I)_* = (p_I)_*.$$

Let $I = \{n\}$. Then

$$\begin{aligned} \mathrm{Ker}(p_n)_* &= \pi_1(\Sigma \setminus \{z_1^0, \ldots, z_{n-1}^0\}, z_n^0) \\ &= \langle \zeta_1, \ldots, \zeta_p, A_{1,n}, \ldots, A_{n-1,n} \rangle \end{aligned}$$

where ζ_i and $A_{r,n}$ are s.c.c.'s based at z_n^0 surrounding i-th puncture and z_r^0 once, respectively.

ϕ is ori. pres. geometric $\Longleftrightarrow$ $\phi|_{\mathrm{Ker}(p_n)_*}$ is ori. pres. geometric

(Lem. 5)

$$\Longleftrightarrow \begin{array}{l} \phi(\zeta_i) = w_i^{-1}\zeta_j w_i, \\ \phi(A_{r,n}) = v_r^{-1}A_{r,n}v_r \\ \text{for some } v_r, w_i\text{'s in } \mathrm{Ker}(p_n)_* \end{array}$$

(Thm. 6)

Nielsen-Thurston normal form I

Let $\bar{f} \in \mathcal{M}(\Sigma, \bar{\mathbf{z}}^0)$. A *reduction system* $\mathbf{C}$ is a finite collection of isotopy classes of simple closed curves in $\Sigma \setminus \bar{\mathbf{z}}^0$ such that

$$\bar{f}(\mathbf{C}) = \mathbf{C}.$$

Then reduction systems form a POSET w. r. t. inclusion.
A *canonical reduction system* $\mathbf{C}(\bar{f})$ is the intersection of all maximal reduction systems for $\bar{f}$.
We say that $\bar{f}$ is :

1. *periodic* if $\bar{f}^N = Id$ for some $N > 0$;
2. *reducible* if $\bar{f}$ has the nonempty canonical reduction system;
3. *pseudo-Anosov* otherwise.

Nielsen-Thurston normal form II

Let T_c be the Dehn twist along c.

Theorem 7 (Nielsen-Thurston normal form)

Let $\mathbf{C}(\bar{f}) = \{c_1, \ldots, c_k\}$ be the canonical reduction system for $\bar{f}$, and $\Sigma \setminus \coprod c_i = \coprod_{i=1}^r \Sigma_i$. Then for some e_i's,

$$\bar{f} = \prod_{i=1}^{r} \bar{f}_i \prod_{i=1}^{k} T_{c_i}^{e_i},$$

where each restriction $\bar{f}_i = \bar{f}|_{\Sigma_i}$ is either periodic or psuedo-Anosov.

Let $\mathrm{Supp}(\bar{f})$ be the set of Σ_i's with pseudo-Anosov $\bar{f}_i$.

Nielsen-Thurston normal form III

Theorem 8 (Ivanov)

Let $\bar{f}$ be a pseudo-Anosov map. Then the centralizer of $\bar{f}$ is virtually infinite cyclic. That is, if $[\bar{f}, \bar{g}] = 1$ for nonperiodic $\bar{g}$, then $\bar{f}^{N_1} = \bar{g}^{N_2}$ for some $N_1, N_2 \neq 0$.

Corollary 9

Let $\bar{f}, \bar{g}$ be two mapping classes. Then $[\bar{f}, \bar{g}] = 1$ only if

1. $\Sigma_i \in \mathrm{Supp}(\bar{f}), \Sigma_j \in \mathrm{Supp}(\bar{g})$ *with* $\Sigma_i \cap \Sigma_j \neq \emptyset$, *then* $\Sigma_i = \Sigma_j$.
2. $\bar{f}^{N_1} = \bar{g}^{N_2}$ *for some* $N_1, N_2 \neq 0$ *on* $\Sigma_i \in \mathrm{Supp}(\bar{f}) \cap \mathrm{Supp}(\bar{g})$.

Proof of the main theorem I

Consider $\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0)$ as a subgroup of $\mathcal{M}(\Sigma, \bar{\mathbf{z}}^0)$.

Theorem 10 (Fadell-Neuwirth)

The braid group $\mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0)$ is torsion-free if $\chi(\Sigma) < 0$.

Lemma 11

For any $\beta \in \mathbf{B}_n(\Sigma, \bar{\mathbf{z}}^0)$, each subsurface in $\mathrm{Supp}(\beta)$ *contains at least one of z_j^0.*

Proof of the main theorem II

Consider a short exact sequence

$$1 \longrightarrow \mathrm{Ker}(p_n)_* \longrightarrow \mathbf{P}_n(\Sigma, \mathbf{z}^0) \xrightarrow{(p_n)_*} \mathbf{P}_{n-1}(\Sigma, p_n(\mathbf{z}^0)) \longrightarrow 1.$$

Let γ be a s.c.c. based at z_n^0 parallel to puncture, and let $N(\gamma)$ be a regular neighborhood of γ.
Let $\Sigma' \overset{\iota}{\hookrightarrow} \Sigma \setminus N(\gamma)$ be a connected component with $\chi(\Sigma') < 0$.
Then there is a homeomorphism $h : (\Sigma, p_n(\mathbf{z}^0)) \to (\Sigma', p_n(\mathbf{z}^0))$ which induces

$$\iota \circ h : F_{n-1}(\Sigma) \simeq F_{n-1}(\Sigma') \hookrightarrow F_n(\Sigma).$$

Proof of the main theorem III

However $\Sigma \setminus \Sigma'$ is 2-punctured disc,

$$s_\gamma = \iota_* \circ h_* : \mathbf{P}_{n-1}(\Sigma, p_n(\mathbf{z}^0)) \to \mathbf{P}_n(\Sigma, \mathbf{z}^0)$$

is an embedding.
Moreover, $(p_n)_* \circ s_\gamma = (p_n \circ \iota \circ h)_* = Id$.
We say that s_γ is a *section defined by* γ.

Lemma 12

Let γ and s_γ be as before. Then

$$[\gamma, \mathrm{Im} s_\gamma] = 1.$$

Furthermore, the centralizer of $\mathrm{Im} s_\gamma$ in $\mathcal{M}(\Sigma, \bar{\mathbf{z}}^0)$ is infinite cyclic generated by γ.

Proof of the main theorem IV

Let $\mathbf{P}_{n-1,1} = \mathrm{Ker}(p_n)_*$.
The diagram below is the simplified version of the diagram of (NC).

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{P}_{n-1,1} & \longrightarrow & \mathbf{P}_n & \underset{s_n}{\overset{(p_n)_*}{\rightleftarrows}} & \mathbf{P}_{n-1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow \phi_n & & \downarrow \phi_{n-1} & & \\
1 & \longrightarrow & \mathbf{P}'_{n-1,1} & \longrightarrow & \mathbf{P}'_n & \underset{s'_n}{\overset{(p'_n)_*}{\rightleftarrows}} & \mathbf{P}'_{n-1} & \longrightarrow & 1
\end{array}$$

Let $s_n : \mathbf{P}_{n-1} \to \mathbf{P}_n$ be a section defined by ζ_i, and
$s'_n : \mathbf{P}'_{n-1} \to \mathbf{P}'_n = \phi_n \circ s_n \circ \phi_{n-1}^{-1}$.

Proof of the main theorem V

By Lemma 12, $[\zeta_i, \mathrm{Im} s_n] = 1$, and so $[\phi_n(\zeta_i), \mathrm{Im} s'_n] = 1$.
By Corollary 9 and Lemma 11, we get for all $\beta' \in \mathbf{P}'_{n-1}$,

$$\mathsf{Supp}(\phi_n(\zeta_i)) \cap \mathsf{Supp}(s'_n(\beta')) = \emptyset.$$

However, since $(p'_n)_* \circ s'_n = Id$ on $\mathbf{P}'_{n-1}$, we get

$$\Sigma_0 = \Sigma \setminus \bigcup_{\beta' \in \mathbf{P}'_{n-1}} \mathsf{Supp}(s'_n(\beta')) \simeq D^2 \setminus \{2pts\}.$$

Hence $\phi_n(\zeta_i) \in \mathcal{M}(\Sigma_0) \simeq \langle T_{\partial \Sigma_0} \rangle$, and so

$$\phi_n(\zeta_i) = T_{\partial \Sigma_0} = w_n^{-1} \zeta_j w_n$$

for some $w_n \in \mathbf{P}'_{n-1,1}$.

Proof of the main theorem VI

Similarly, we can show that for any $1 \leq r < n$,

$$\phi_n(A_{r,n}) = v_r^{-1} A_{r,n} v_r$$

for some $v_r \in \mathbf{P}'_{n-1,1}$.

$\square$