Involutions on spin 4-manifolds

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Abstract

In this talk, some topics and results around nonsmoothable group actions on spin 4-manifolds are given. Especially, I will explain our recent result about nonsmoothable $\mathbb{Z}_2$-actions on spin 4-manifolds.

Contents

1. Introduction

Let X be an oriented topological manifold and G a finite group.

A topological G-action on X is called locally linear if $\forall x \in X$, $\exists V_x$: a G_x-invariant neighborhood of X, (G_x: isotropy subgroup of x) such that

(1) $V_x \cong \mathbb{R}^n$ (homeomorphism),

(2) G_x acts on $\mathbb{R}^n \cong V_x$ in a linear orthogonal way.

It is well-known that every smooth action is locally linear, but locally linear action is not necessary to be smooth.

A locally linear action of a group on a topological manifold X is called nonsmoothable if the action is not smooth with respect to any possible smooth structure on X.

In recent years, many nonsmoothable group actions on 4-manifolds are constructed by many authors, for examples, Kwasik and Lee (1988), Kwasik and Lawson (1993), Hambleton-Lee (1995), Hambleton and Tanase (2004), Liu and Nakamura (2008) and Nakamura (2009).

Recently, Kiyono [Algeb. and Geom. Topol., **11**(2011), 1345–1359] constructed a homologically trivial, pseudofree, locally linear action of $\mathbb{Z}_p$ on spin 4-manifolds for any sufficiently large prime p which is nonsmoothable. That is he proved the following result.

Theorem 1 [Kiyono] *Let X be a closed, simply connected, spin topological 4-manifold not homeomorphic to either S^4 or $S^2 \times S^2$. Then, for any sufficiently large prime number p, there exists a homologically trivial, pseudofree, locally linear action of $\mathbb{Z}_p$ on X which is nonsmoothable.*

An action is called pseudofree if it is free on the compliment of a discrete subset.

In this talk we consider the minimal order $p = 2$. We have constructed nonsmoothable locally linear pseudofree $\mathbb{Z}_2$-actions on a large class of spin 4-manifolds. That is we proved the following main result:

Theorem 2 *Let X be a closed, simply-connected, smooth, spin 4-manifold whose intersection form is isomorphic to $n(-E_8) \bigoplus mH$, where H is the hyperbolic form. If $n \equiv 2 \mod 4$, then there exists a locally linear pseudofree $\mathbb{Z}_2$-action on X which is nonsmoothable with respect to any possible smooth structure on X.*

How to construct a nonsmoothable action?

Usually two steps:

(1) Existence: To construct loc. lin. actions concretely.
(2) Nonsmoothable: Prove the above action is nonsmoothable.

When $G = \mathbb{Z}_p$, p is a prime, X is a closed simply connected 4-manifold, general construction for (1) is established by Edmonds-Ewing.

(i) fixed point data
(ii) G-action on the intersection form
with certain conditions $\Rightarrow$ $\exists$ locally linear action realizing (i) (ii).

For (2) various techniques are used. Most of the results use gauge theory to prove the nonsmoothability. Kiyono uses Seiberg-Witten theory (G-equiv 10/8-inequality) to prove the nonsmoothability.

$G \curvearrowright X \Rightarrow$ quotient V-manifold X/G should satisfy 10/8-type inequality.

The proof of our main result is divided into two steps. In the first step, we give a constraint on smooth involutions. In the second step, we construct a locally linear action which would violate the constraint. To obtain a constraint on smooth involutions, we do not use gauge theory, but only use Rohlin's theorem.

2. Preliminaries

The purpose of this section is to prepare some tools used later.

Let X be a closed oriented smooth 4-manifold, we know it admits a fundamental class $[X] \in H_4(X; Z)$.

Definition 2.1 The symmetric bilinear form

$$Q_X : H^2(X; Z) \times H^2(X; Z) \to Z$$

defined by $Q_X(a, b) =< a \cup b, [X] >= a \cdot b \in Z$ is called the intersection form of X.

Definition 2.2 (a) For a given symmetric bilinear form Q on a finitely generated free abelian group A. The number of $+1$'s (-1's resp.) on the diagonal is denoted by b_2^+ (resp. b_2^-); the difference $b_2^+ - b_2^-$ is the signature $\sigma(Q)$. Finally Q is even is $Q(a, a) \equiv 0(\mathrm{mod}2)$ for every $a \in A$; Q is odd otherwise.

(b) Q is positive (negative) definite if $\mathrm{rk}(Q) = \sigma(Q)$ ($\mathrm{rk}(Q) = -\sigma(Q)$ resp.). Q is indefinite otherwise.

(c) Q is called unimodular if $\det Q = \pm 1$.

Consider the following 8×8 intersection form matrix E_8:

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}$$

As a matrix of a bilinear form Q on Z^8, E_8 gives a definite, even, unimodular form with $\sigma(Q) = 8$.

Denote by H the hyperbolic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Using E_8 and H, we can build up many indefinite unimodular forms $Q = aE_8 \oplus bH$, where $a = \frac{\sigma(Q)}{8}$ and $b = \frac{\mathrm{rk}(Q)-|\sigma|}{2}$.

Theorem 2.1 *Suppose that Q is an indefinite, unimodular form. If Q is odd, then it is isomorphic to $b_2^+(1) \oplus b_2^-(-1)$; if Q is even, then it is isomorphic to $\frac{\sigma(Q)}{8}E_8 \oplus \frac{\mathrm{rk}(Q)-|\sigma(Q)|}{2}H$.*

Theorem 2.2 (Rohlin) *If the intersection form Q_X of a smooth, simply connected, closed 4-manifold X is even, then the signature $\sigma(X)$ is divisible by 16.*

From the classification of unimodular even integral quadratic forms and the Rochlin's theorem, for the choice of orientation with non-positive signature the intersection form of a closed spin 4-manifold X is

$$-2kE_8 \oplus mH, \qquad k \geq 0.$$

In 1995, by using the finite dimensional approximation of the Seiberg-Witten equations, Furuta proved:

$\frac{10}{8}$-inequality:

$$b_2(X) \geq \frac{10}{8}|\sigma(X)| + 2. \tag{2.1}$$

It is equivalent to the following:

Theorem 2.3 (Furuta) *Let X be a smooth spin 4-manifold with $b_1(X) = 0$ with non-positive signature. If $m \neq 0$, then*

$$2k + 1 \leq m.$$

3. Construction of locally linear $\mathbb{Z}_2$-actions

To construct locally linear $\mathbb{Z}_2$-actions, we use the realization theorem by Edmonds and Ewing. For our purpose, we summarize their result in a very special case.

Theorem 3.1 [Edmonds and Ewing (1992)] *Let* $\Psi : V \times V \to \mathbb{Z}$ *a* $\mathbb{Z}_2$*-inv. symm. unimodular even form s.t.*

(1) As a $\mathbb{Z}[\mathbb{Z}_2]$*-module,* $V \cong F \oplus T$*, where* F *is free* $\mathbb{Z}[\mathbb{Z}_2]$*-module and* T *is a trivial* $\mathbb{Z}[\mathbb{Z}_2]$*-module with* $\operatorname{rank}_{\mathbb{Z}} T = n$*.*

(2) For any $v \in V$, $\Psi(gv, v) \equiv 0 \mod 2$.

(3) G-signature formula is satisfied, i.e., $\operatorname{Sign}(g, (V, \Psi)) = 0$.

then there exists a locally linear, pseudofree, $\mathbb{Z}_2$*-action on simply-connected 4-manifold* X *s.t.*

(i) Its intersection form is Ψ*.*

(ii) $\sharp X^{\mathbb{Z}_2} = n + 2$.

Remark: Since Ψ is supposed to be even, the homeomorphism type of X is unique by Freedman's theorem.

For our application, we need to recall their equivariant handle construction precisely.

Let B_0 be a unit ball in $\mathbb{C}^2$, $\mathbb{Z}_2$ acts on it by multiplication of ± 1. Take a $\mathbb{Z}_2$-invariant knot K in $S_0 = \partial B_0$. Then we can represent a framing of K by an equivariant embedding $S^1 \times D^2 \to S_0$ for some $\mathbb{Z}_2$ action on $S^1 \times D^2$. In particular, 0-framing is represented by $f_0 : S^1 \times D^2 \to S_0$ such that $f_0(S^1 \times \{0\}) = K$, and $f_0(S^1 \times \{1\})$ has linking number 0 with K, and the $\mathbb{Z}_2$ action on $S^1 \times D^2 \subset \mathbb{C}^2$ is given by $g(z, w) = (-z, -w)$. An arbitrary r-framing of K can be represented by $f_r : S^1 \times D^2 \to S_0$ given by $f_r(z, w) = f_0(z, z^r w)$. Then f_r is equivariant if the $\mathbb{Z}_2$ on $S^1 \times D^2$ is given by $g(z, w) = (-z, (-1)^{r-1} w)$.

For a given K and a framing r, we can construct a 4-manifold with a $\mathbb{Z}_2$-action as $W = B_0 \cup_{f_r} D^2 \times D^2$.

Let $H_1, \ldots, H_n$ be copies of $D^2 \times D^2$ on which $\mathbb{Z}_2$ acts by $g(z, w) = (-z, -w)$. Note that if the framing r is even, then we can attach H_i to B_0 equivariantly via f_r.

Edmonds-Ewing's construction of locally linear actions is divided into three steps.

Step 1. Represent Ψ by a $\mathbb{Z}_2$-invariant n-component framed link L in ∂B_0.

Under the assumption of Theorem 3.1, we may assume $\Psi|_T$ is represented by a matrix (a_{ij}) such that a_{ii} is even and a_{ij} is odd whenever $i \neq j$. Actually, by changing the basis of T, we can take a n-component link L_T in $S_0 = \partial B_0$, such that L_T component representing the matrix (a_{ij}). Thus $\Psi|_T$ is represented by a framed link L_T in S_0, and it is not difficult to realize the other part of Ψ by a link, and therefore we obtain a framed link L in S_0 which realizes the given $\mathbb{Z}_2$-invariant form Ψ.

Step 2. Attach $H_1, \dots, H_n$ and free 2-handles to B_0 along L equivariantly. We can do this since the diagonal entries of (a_{ij}) are even. Thus we obtain a 4-manifold X_0 on which $\mathbb{Z}_2$ acts smoothly,

$$X_0 = B_0 \cup H_1 \cup \cdots \cup H_n \cup (\text{free handles}).$$

Step 3. The boundary of X_0 is an integral homology 3-sphere Σ with a free $\mathbb{Z}_2$-action. Under the assumption of Theorem 3.1, Edmonds and Ewing proved that there exists a contractible 4-manifold W with a locally linear $\mathbb{Z}_2$-action such that its boundary is Σ with the given free $\mathbb{Z}_2$-action, and it has exact on fixed point. Then we obtain the required manifold $X = X_0 \cup W$ with the required action.

Each of B_0, H_1, $\cdots$, H_n, W has one fixed point, denoted by $P, Q_1, \cdots, Q_n, P'$. The action constructed above is smooth on X_0, and is smooth on X except near the final fixed point.

4. Atiyah-Bott's criterion for $\varepsilon(P)$

For a smooth even-type (pseudofree equivalently) $\mathbb{Z}_2$-action on a simply-connected smooth spin 4-manifold X, the spin structure on X is unique up to equivalence. Therefore every involution on X preserves the spin structure and also the Spin^c-structure c_0 which is determined by the spin structure. The sign assignment (introduced by Atiyah and Bott) determined by the lift of the $\mathbb{Z}_2$-action to c_0 is $\varepsilon : X^{\mathbb{Z}_2} \to \pm 1$.

Let g be the generator of the $\mathbb{Z}_2$-action, by the G-spin theorem, we have

$$\mathrm{ind}_g D = k_+ - k_- = \frac{1}{4} \sum_{P \in X^{\mathbb{Z}_2}} \varepsilon(P),$$

$$\mathrm{ind} D = k_+ + k_- = -\frac{1}{8}\sigma(X),$$

where k_+ and k_- are coefficients of the $\mathbb{Z}_2$-index of the Dirac operator. Note that k_+ and k_- are even because of the quartanionic structure of Dirac index. Then the sum $\sum_{P \in X^{\mathbb{Z}_2}} \varepsilon(P)$ is a multiple of 8 by the above two equations.

Suppose $g \in \mathbb{Z}_2$ is nontrivial element, we can lift the smooth pseudofree involution $g : X \to X$ to the frame bundle F as $g_* : F \to F$ if a g-invariant metric is fixed. A spin structure on X if given by a double cover $\varphi : \hat{F} \to F$, where $\hat{F}$ is a Spin(4) bundle.

The values $\varepsilon(P)$ and $\varepsilon(Q)$ of distinct fixed points P and Q can be compared by Atiyah-Bott's criterion as following proposition. We take a path s in F starting form a point $y \in F_P$ and ending at $y^{'} \in F_Q$. Then the path $-g_* s$ has the same starting point and the end point as s, where " $-$ " means the multiplication by -1 on each fiber. Thus by connecting s and $-g_* s$, we obtain a circle C in F.

Proposition 4.1 *The preimage $\varphi^{-1}(C)$ has two components if and only if $\varepsilon(P) = \varepsilon(Q)$. In other words, the preimage $\varphi^{-1}(C)$ is connected if and only if $\varepsilon(P) = -\varepsilon(Q)$.*

Recall that each component of B_0 and H_i of X_0 constructed section 3 has a fixed point, denoted by P and Q_i, suppose that H_i is attached to B_0 equivariantly along a knot K with a framing r. Then we have the following proposition.

Proposition 4.2 *Suppose K is a trivial knot in ∂B_0 which bounds a $\mathbb{Z}_2$-invariant embedded disk D_0 in B_0 containing P. If $r \equiv 2 \mod 4$, then $\varepsilon(P) = \varepsilon(Q)$. If $r \equiv 0 \mod 4$, then $\varepsilon(P) = -\varepsilon(Q)$.*

Let X be an oriented topological manifold and G a finite group. The sign assignment can be defined for locally linear actions by using Atiyah-Bott's criterion on topological spin structure $\varphi : \hat{F} \to F$. On the fiber of F over each fixed point P, there is a point y_P which is mapped to $-y_P$ by the $\mathbb{Z}_2$-action. For distinct fixed points P and Q, by taking a path s connecting such a y_P with such y_Q, we can define the sign assignment as follows.

Definition 4.1 *For each pair (P, Q) of fixed points, let s be a path in F as above, and C the circle formed by s and $-g_*s$. Define $\varepsilon'(P,Q)$ by $\varepsilon'(P,Q) = 1$, if $\varphi^{-1}(C)$ has 2 components; $\varepsilon'(P,Q) = -1$, if $\varphi^{-1}(C)$ is connected.*

Note that this definition does not depend on smooth structures. Furthermore, if the action is realized by a smooth action, then

$$\varepsilon'(P,Q) = \varepsilon(P)\varepsilon(Q). \tag{4.1}$$

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5. Constraint on smooth involutions

Let X be a smooth, closed, oriented, simply-connected, spin 4-manifold. Suppose that $\mathbb{Z}_2$ acts on X smoothly, pseudofreely in an orientation-preserving way. We can give a constraint on smooth involutions by considering Rohlin's theorem.

$X^{\mathbb{Z}_2}$ is discrete $\Leftrightarrow$ the $\mathbb{Z}_2$-action lifts to the spin structure. $\Rightarrow$ $X/\mathbb{Z}_2$ is a spin V-manifold.

Remark. Quotient singularities are cones of $\mathbb{R}P^3$.

Some facts:

(1) There are 2-equivalence classes $s_\pm$ of spin structures on $\mathbb{R}P^3$.

(2) Let $\tilde{s}_\pm$ be the unique spin structure on the disk bundle $D_\pm$ over S^2 of degree ± 2, then $s_\pm = \tilde{s}_\pm|_{\partial D_\pm}$.

(3) Define the spin type of fixed point by the spin structure on $\mathbb{R}P^3$ induced from $X/\mathbb{Z}_2$.

(4) Let $n_\pm = \sharp$ (fixed points corresponding to $s_\pm$), so $\sharp X^{\mathbb{Z}_2} = n_+ + n_-$.

We know that the quotient space $X/\mathbb{Z}_2$ is not a smooth 4-manifold. In order to use Rohlin's theorem, we need to make $X/\mathbb{Z}_2$ to be smooth. Since $\mathbb{R}P^3$ has two equivalent classes of spin structures, we need to make the spin structures are compatible. Remove cones of $\mathbb{R}P^3$ from $X/\mathbb{Z}_2$, and glue disk bundles D_+ and D_- so that spin structures are compatible. We get a smooth spin 4-manifold, and apply Rohlin's theorem, we have

$$\sigma(X/\mathbb{Z}_2) \equiv n_+ - n_- \text{mod } 16.$$

The G-signature theorem is

$$\sigma(X/\mathbb{Z}_2) = \frac{1}{2}\sigma(X),$$

that is

$$\frac{1}{2}\sigma(X) \equiv n_+ - n_- \mathrm{mod}\ 16.$$

So for a smooth $\mathbb{Z}_2$-action on X as above, we have

$$\begin{cases} \sharp X^{\mathbb{Z}_2} = n_+ + n_-, \\ \frac{1}{2}\sigma(X) \equiv n_+ - n_- \mathrm{mod}\ 16. \end{cases} \tag{5.1}$$

Remark: The numbers n_+ and n_- do not depend on smooth structures, i.e. they are invariants of locally linear pseudofree involutions on topological spin 4-manifolds.

There is a relation between the spin types and the sign assignments of two fixed points.

Proposition 5.1 *Let P and Q be distinct fixed points. Then, $\varepsilon(P) = \varepsilon(Q)$ iff P and Q have the same spin type; $\varepsilon(P) = -\varepsilon(Q)$ iff P and Q have the different spin types.*

For the fixed points obtained in section 3, we can compare their spin types by the following proposition.

Proposition 5.2 *Suppose* K_i *bounds a* $\mathbb{Z}_2$*-invariant embedded disk in* B_0*, and* r_i *is the framing of* K_i*. Then,* $r_i \equiv 2 \mod 4$ *if and only if* P *and* Q_i *have the same spin types;* $r_i \equiv 0 \mod 4$ *if and only if* P *and* Q_i *have the different spin types.*

6. Proof of the main result

Let X be a closed, simply-connected, smooth, spin 4-manifold which has the intersection form isomorphic to $n(-E_8) \bigoplus mH$. Therefore $m \geq n+1$ by Furuta's $\frac{10}{8}$-inequality. Suppose an orientation-preserving smooth pseudofree $\mathbb{Z}_2$-action on X is given.

Since $\sigma(X) = -8n$, so $-4n \equiv \ n_+ - n_-$ mod 16. Therefore, if $n_+ = n_-$, then $n \equiv \ 0$ mod 4.

Now we can construct a locally linear $\mathbb{Z}_2$-action on X. By the realization theorem by Edmonds and Ewing, if we fix an appropriate $\mathbb{Z}_2$-action on the intersection form, then we have a locally linear $\mathbb{Z}_2$-action on X.

Define $\mathbb{Z}_2$-action on $\Psi = n(-E_8) \bigoplus mH$ as follows. Let $\mathbb{Z}_2$ act on a $\frac{m-r}{2}H \bigoplus \frac{m-r}{2}H$ summand by permutation of two $\frac{m-r}{2}H$'s, where r is an integer less than m and such that $r \equiv m \mod 2$. Similarly, let $\mathbb{Z}_2$ act on a $n(-E_8) = \frac{n}{2}(-E_8) \bigoplus \frac{n}{2}(-E_8)$ summand by permutation of two $\frac{n}{2}(-E_8)$'s, and on the rest rH trivially. The trivial part is denoted by T.

Now consider the matrix

$$A = \begin{pmatrix} 0 & \cdots & 1 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 1 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 2 & \cdots & 1 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 & \cdots & 2 \end{pmatrix},$$

which is a $2r \times 2r$ matrix such that the first r diagonal entries are 0, and the rest r diagonal entries are 2, and all of off-diagonal entries are 1.

This symmetric form is even and indefinite. By diagonalizing the matrix A, we can see that the numbers of $+1$ and -1 on the diagonal are equal. The determinant of A is $(-1)^r$. So the symmetric form represented by A is isomorphic to rH.

Hence, we may assume $\Psi|_T$ is represented by the matrix A. Furthermore, the matrix A can be realized by a link whose every component bounds a $\mathbb{Z}_2$-invariant embedded disk. By equivariant handle construction, we can construct a smooth action on a manifold X_0 with boundary, and this action can be extended to the whole X as a locally linear action.

So we have constructed a pseudofree, locally linear $\mathbb{Z}_2$-action on X which satisfies

$$\sharp X^{\mathbb{Z}_2} = 2r + 2.$$

Denote the fixed points by P, $Q_1, \cdots, Q_{2r}$, P' as above. Recall that the $\mathbb{Z}_2$-action we have constructed is smooth except near P'. By Proposition 3.2, a half number of fixed points in $\bigcup_{1\leq i\leq 2r} Q_i$ have the same spin type with P, and the rest fixed points in $\bigcup_{1\leq i\leq 2r} Q_i$ have different spin type with P. If the $\mathbb{Z}_2$-action can be smooth on X, by the fact that $\sum_{P\in X^{\mathbb{Z}_2}} \varepsilon(P)$ is a multiple of 8, we have $\sum_{P\in X^{\mathbb{Z}_2}} \varepsilon(P) = 0$. Thus, P and P' have the different spin types. So we have

$$(n_+, n_-) = (r+1, r+1).$$

But for X whose intersection form is isomorphic to $n(-E_8)\bigoplus mH$ with $n \equiv 2 \bmod 4$, we can not have $n_+ = n_-$. Thus the main result is proved.

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