The zeroth coefficient polynomial of a $(2,1)$-cable knot

Hideo Takioka (M2, Osaka City University)

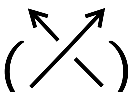
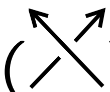
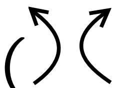
Contents

§1. The zeroth coefficient polynomial of the HOMFLYPT polynomial

L: a link.

$^{\exists}1\, P(L) = P(L; y, z) \in \mathbb{Z}[y, y^{-1}, z, z^{-1}]$: a link invariant s.t.

(i) $P(\bigcirc) = 1$.

(ii) $yP($$) + y^{-1}P($$) = zP($$)$.

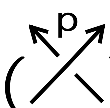
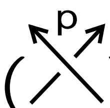
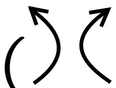
L: an r-component link.

$$P(L; y, z) = (yz)^{-r+1} \sum_{n \geq 0} c_n(L; -y^2) z^{2n}.$$

We call $c_n(L) = c_n(L; x)$ ($x := -y^2$) the n-th coefficient polynomial of the HOMFLYPT polynomial. ($c_{-1}(L) := 0$)

The skein relation is given by Kawauchi as follows:

(i) $c_n(\bigcirc) = \begin{cases} 1 & (n = 0), \\ 0 & (n \neq 0). \end{cases}$

(ii) $-xc_n($$) + c_n($$) = (-x)^{\delta(p)} c_{n-\delta(p)}($$)$,

where $\delta(p) := \begin{cases} 0 & (p\text{: self-crossing}), \\ 1 & (p\text{: non-self-crossing}). \end{cases}$

A property of $c_0(L)$

$L = K_1 \cup \cdots \cup K_r$: an r-component link.

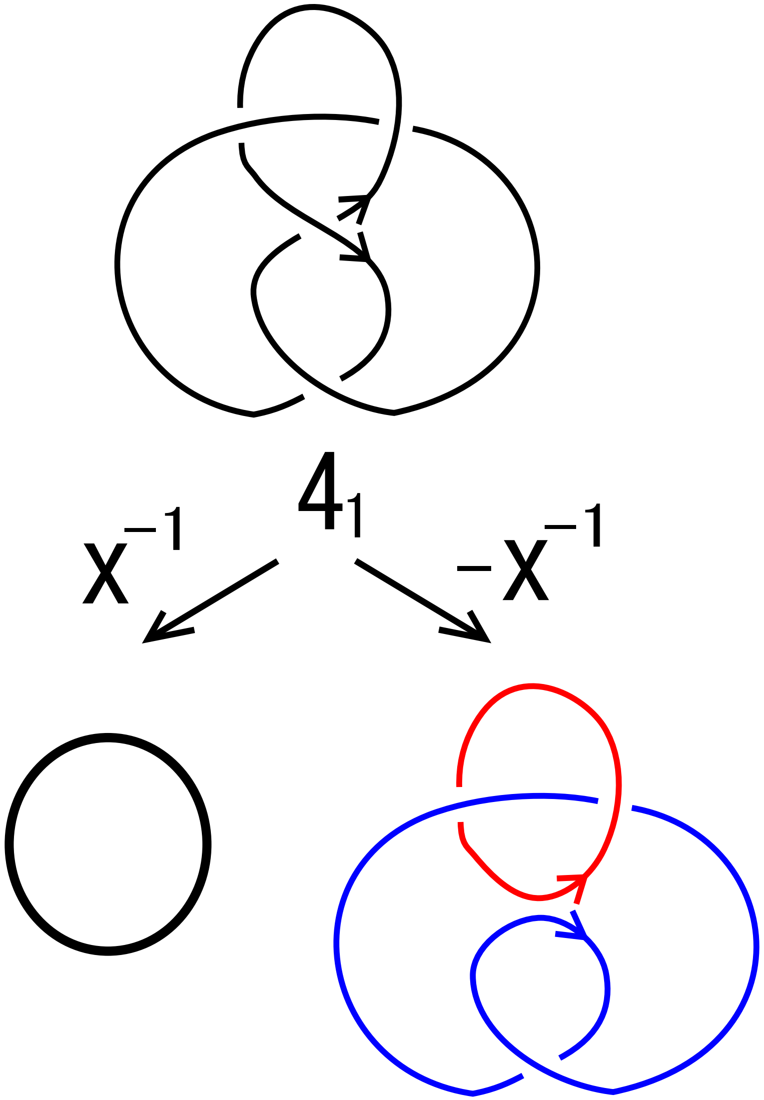
$\mathrm{Link}(L) := \sum_{1 \leq i < j \leq r} \mathrm{Link}(K_i, K_j)$.

Then,

$$c_0(L) = (1 - x)^{r-1} x^{-\mathrm{Link}(L)} c_0(K_1) \cdots c_0(K_r).$$

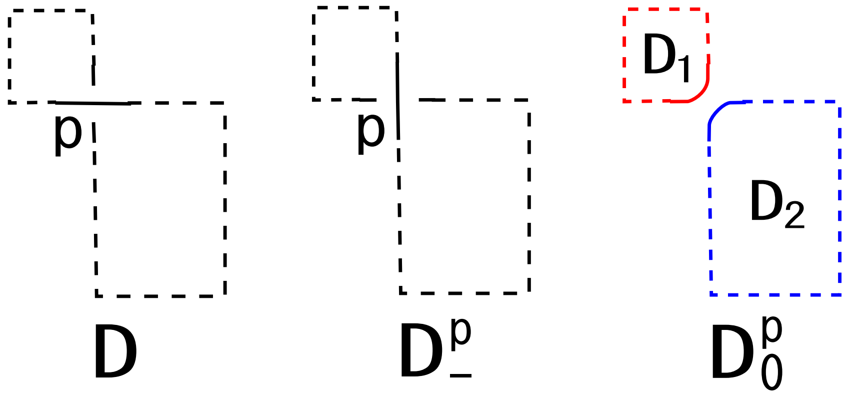
Example

$$c_0(4_1) = x^{-1} - x^{-1}(1-x)x = x + x^{-1} - 1.$$

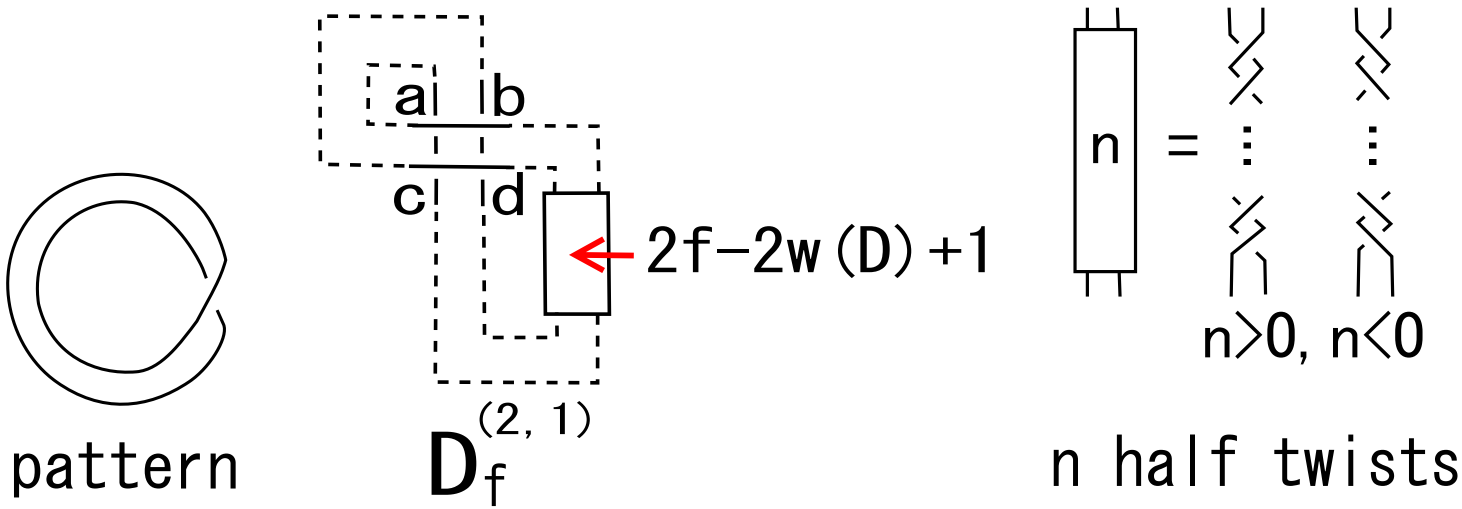


§2. A $(2,1)$-cable knot diagram

Notation

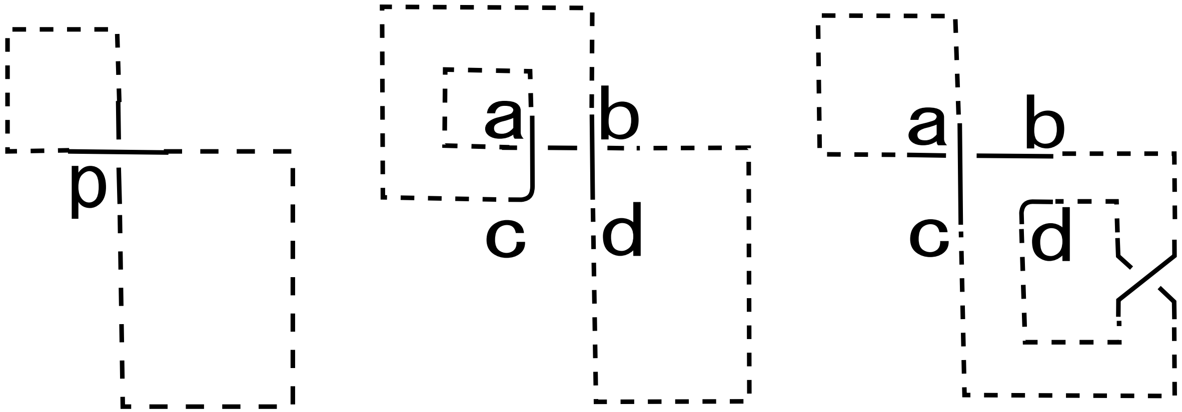


A $(2,1)$-cable knot diagram $D_f^{(2,1)}$ with a framing f of the diagram D is defined by the following diagram. (If $f = 0$, then we write $D^{(2,1)}$ instead of $D_0^{(2,1)}$.)



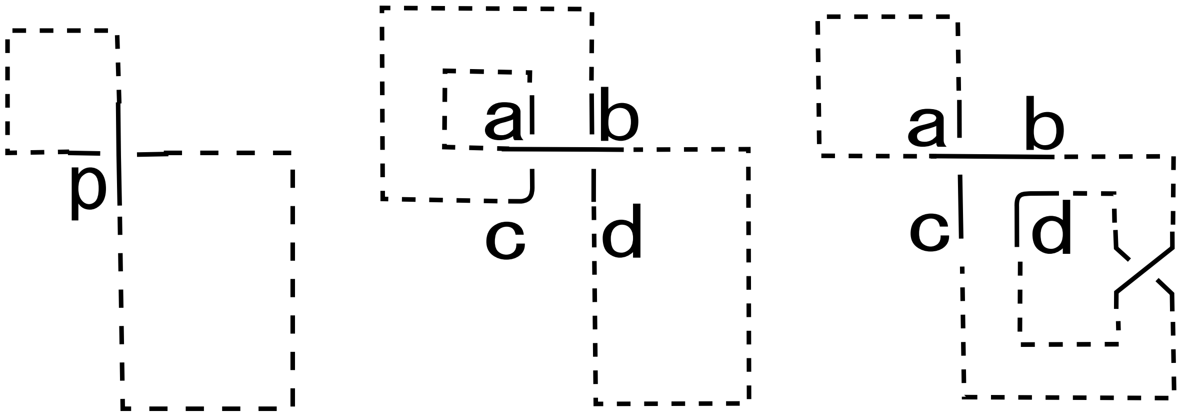
$\varepsilon = \varepsilon(p)$: the sign of a crossing p.

$\varepsilon = +1$



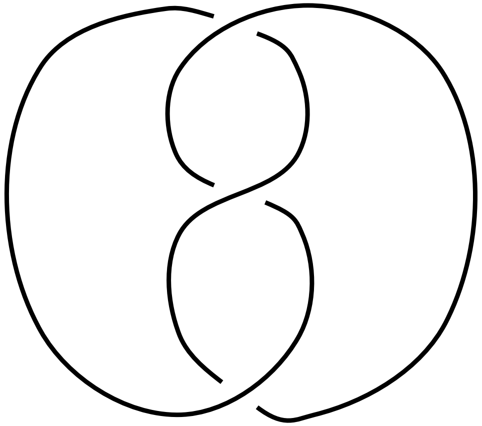
D $\quad$ $\mathbf{E_1^+}$ $\quad$ $\mathbf{E_2^+}$

$\varepsilon = -1$

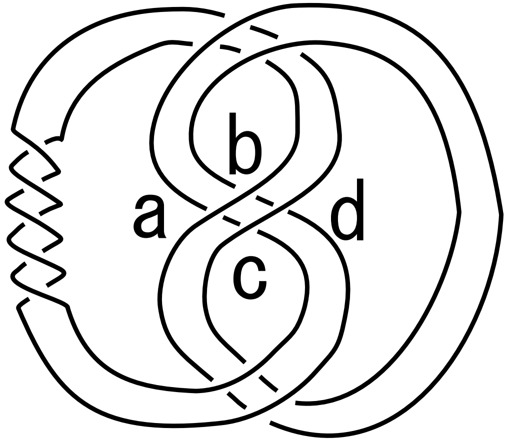


D $\quad$ $\mathbf{E_1^-}$ $\quad$ $\mathbf{E_2^-}$

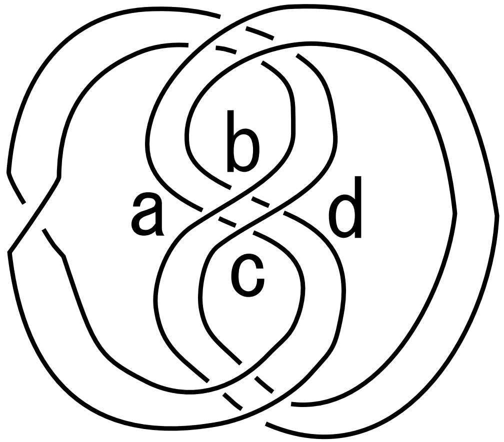
Example



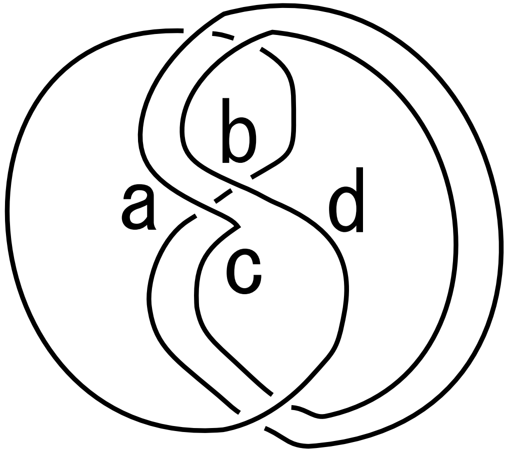
D



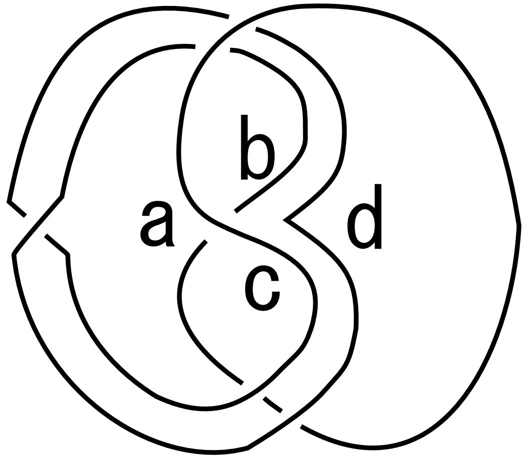
$D_0^{(2,1)} = D^{(2,1)}$



$D_3^{(2,1)}$



E_1^+



E_2^+

Theorem

$$c_0(D^{(2,1)})$$

$$= x^{w(D)-4\varepsilon} c_0(D^{(2,1)\,a\,b\,c\,d}_{w(D)----})$$

$$-2\varepsilon(1-x)x^{w(D_1)-2\varepsilon-1} c_0(E_1^{\varepsilon}) c_0(D_2)$$

$$-2\varepsilon(1-x)x^{w(D_2)-\frac{3}{2}\varepsilon-\frac{1}{2}} c_0(D_1) c_0(E_2^{\varepsilon})$$

$$+(1-x)^2 x^{-\,\mathrm{Link}(D_1,D_2)-\frac{3}{2}\varepsilon-\frac{3}{2}} c_0(D_1) c_0(D_2) c_0(D^p_-)$$

$$+(1-x)^2 x^{-2\,\mathrm{Link}(D_1,D_2)+w(D_2)-1} c_0(D_1)^2 c_0({D_2}^{(2,1)}_{w(D_2)+\varepsilon})$$

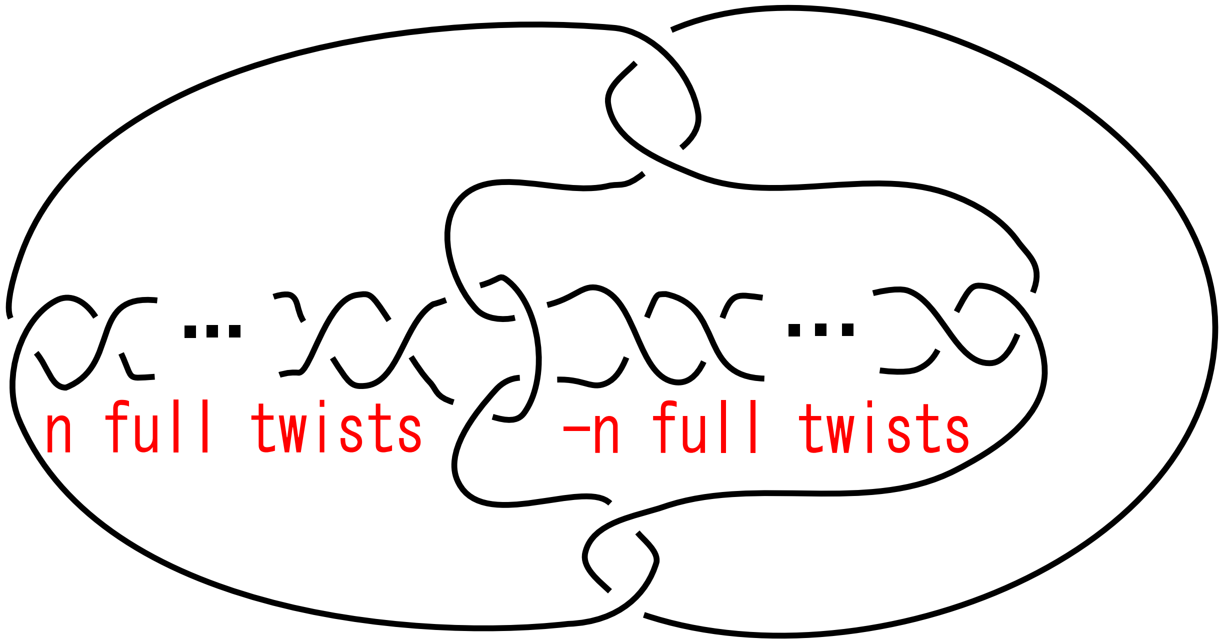
$$+(1-x)^2 x^{-2\,\mathrm{Link}(D_1,D_2)+w(D_1)-\frac{3}{2}\varepsilon-\frac{3}{2}} c_0({D_1}^{(2,1)}_{w(D_1)-\frac{1}{2}\varepsilon-\frac{1}{2}}) c_0(D_2)^2$$

$$+w(D)(1-x)x^{-1} c_0(D)^2.$$

§3. Main Theorem

It is known that the Kanenobu knots $k(n)$ $(n = 0, 1, 2, \ldots)$ have the same HOMFLYPT polynomial:

$$P(k(n)) = (y^2 + y^{-2} + 1 - z^2)^2.$$



$k(n)$

Theorem

$$c_0(k(0)^{(2,1)})$$

$$= 5x^4 - 22x^3 + 48x^2 - 60x + 39 + 4x^{-1} - 34x^{-2} + 34x^{-3}$$

$$-17x^{-4} + 4x^{-5}.$$

$$c_0(k(n)^{(2,1)}) - c_0(k(n-1)^{(2,1)}) \ (n \geq 1)$$

$$= -2x^{n+2} + 8x^{n+1} - 10x^n + 10x^{n-2} - 8x^{n-3} + 2x^{n-4}$$

$$+2x^{-n+3} - 8x^{-n+2} + 10x^{-n+1} - 10x^{-n-1} + 8x^{-n-2} - 2x^{-n-3}.$$

The Morton-Franks-Williams inequality

L: a link.

braid(L): the braid index of L.

y- span $P(L; y, z) :=$ y- maxdeg $P(L; y, z) -$ y- mindeg $P(L; y, z)$.

span $c_0(L) :=$ maxdeg $c_0(L) -$ mindeg $c_0(L)$.

$$\frac{1}{2} y\text{- span}\, P(L; y, z) + 1 \leq \text{braid}(L).$$

$$\text{span}\, c_0(L) + 1 \leq \text{braid}(L).$$

Corollary

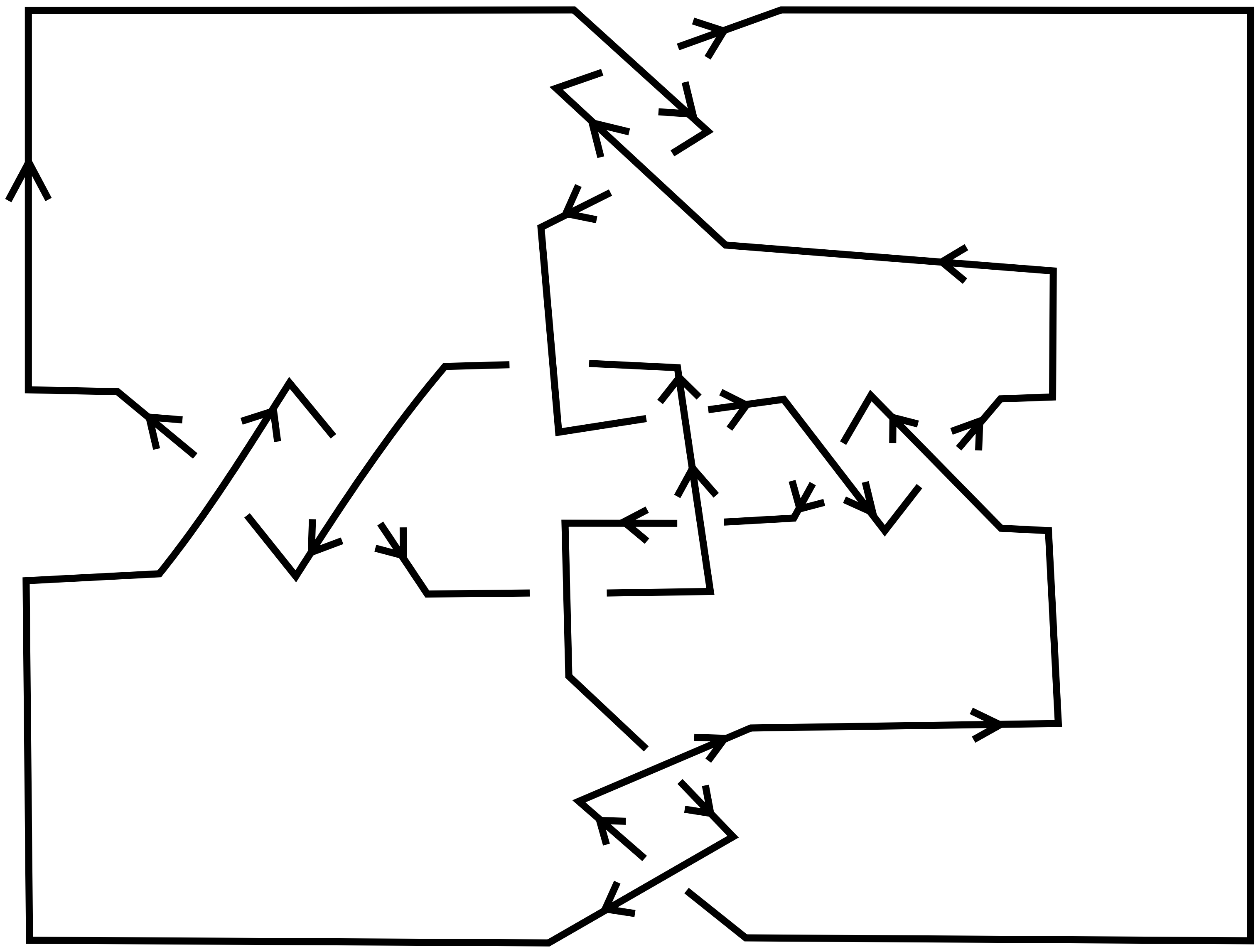
$$\begin{cases} 5 & (n=0,1,2), \\ n+3 & (n \geq 3) \end{cases} \leq \mathrm{braid}(k(n)) \leq \begin{cases} 5 & (n=0,1), \\ 2n+3 & (n \geq 2). \end{cases}$$

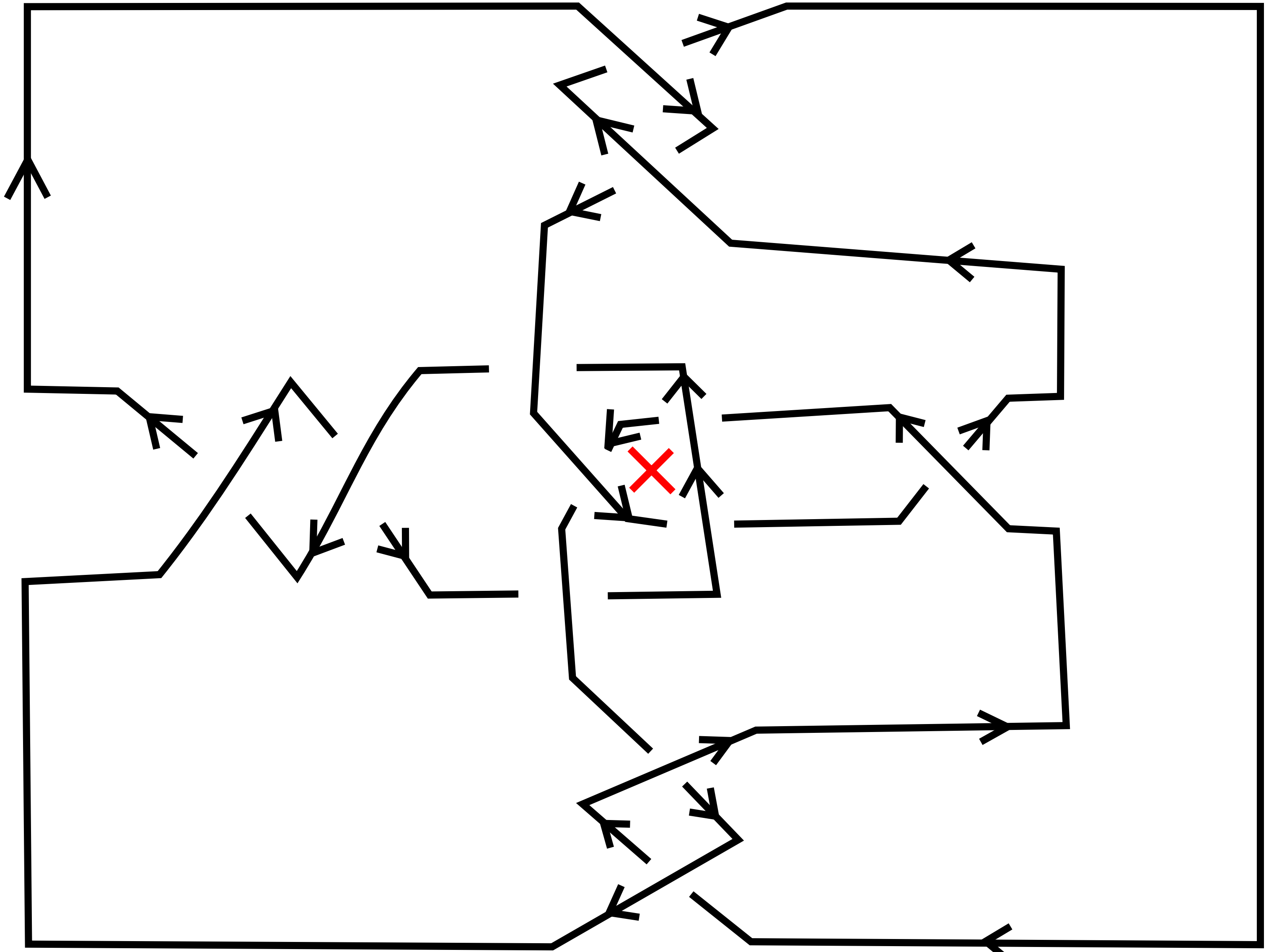
- **a lower bound**

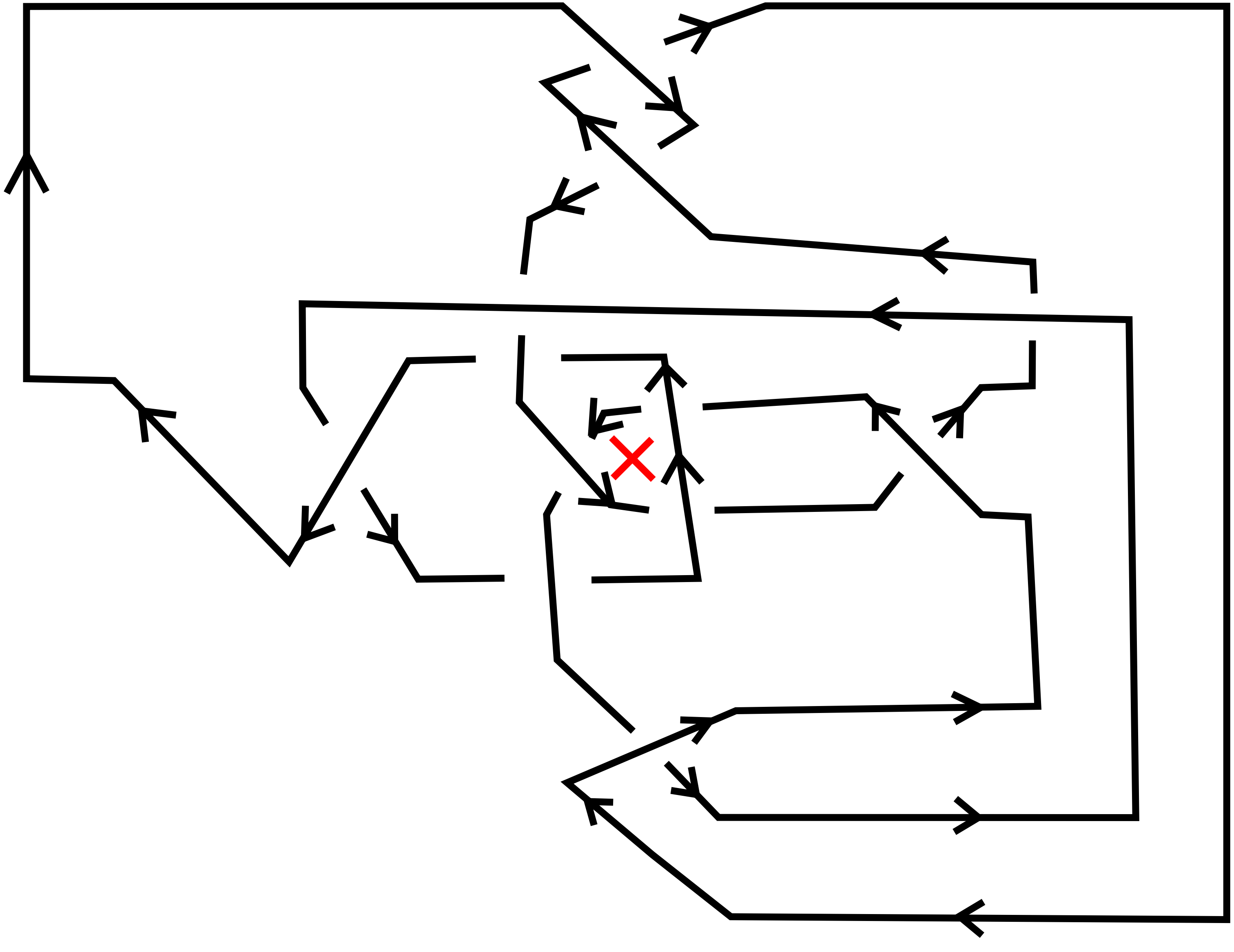
$$\mathrm{span}\, c_0(k(n)^{(2,1)}) + 1 \leq \mathrm{braid}(k(n)^{(2,1)}) \leq 2\,\mathrm{braid}(k(n)).$$

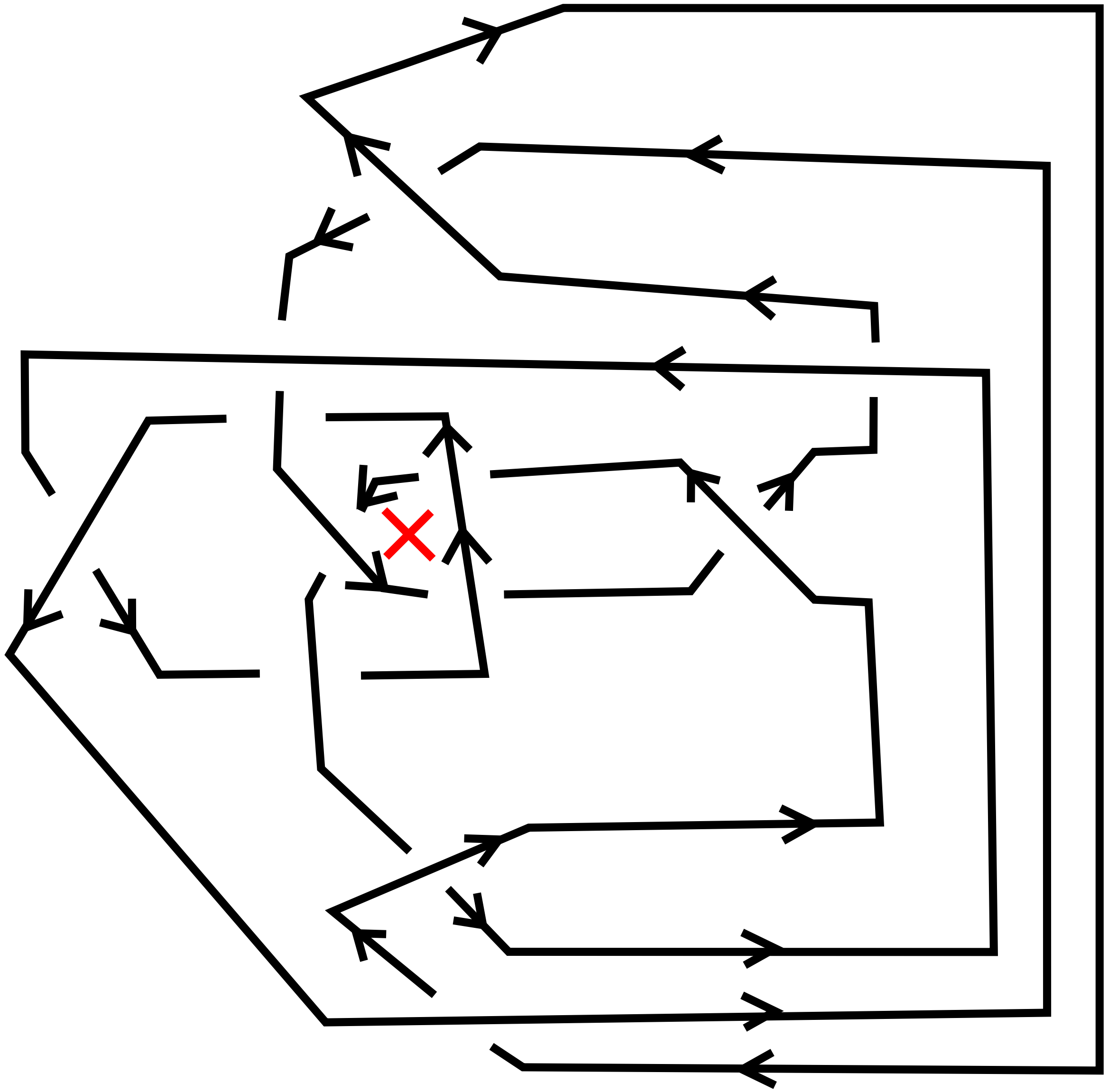
$$\mathrm{span}\, c_0(k(n)^{(2,1)}) = \begin{cases} 9 & (n=0,1,2), \\ 2n+5 & (n \geq 3). \end{cases}$$

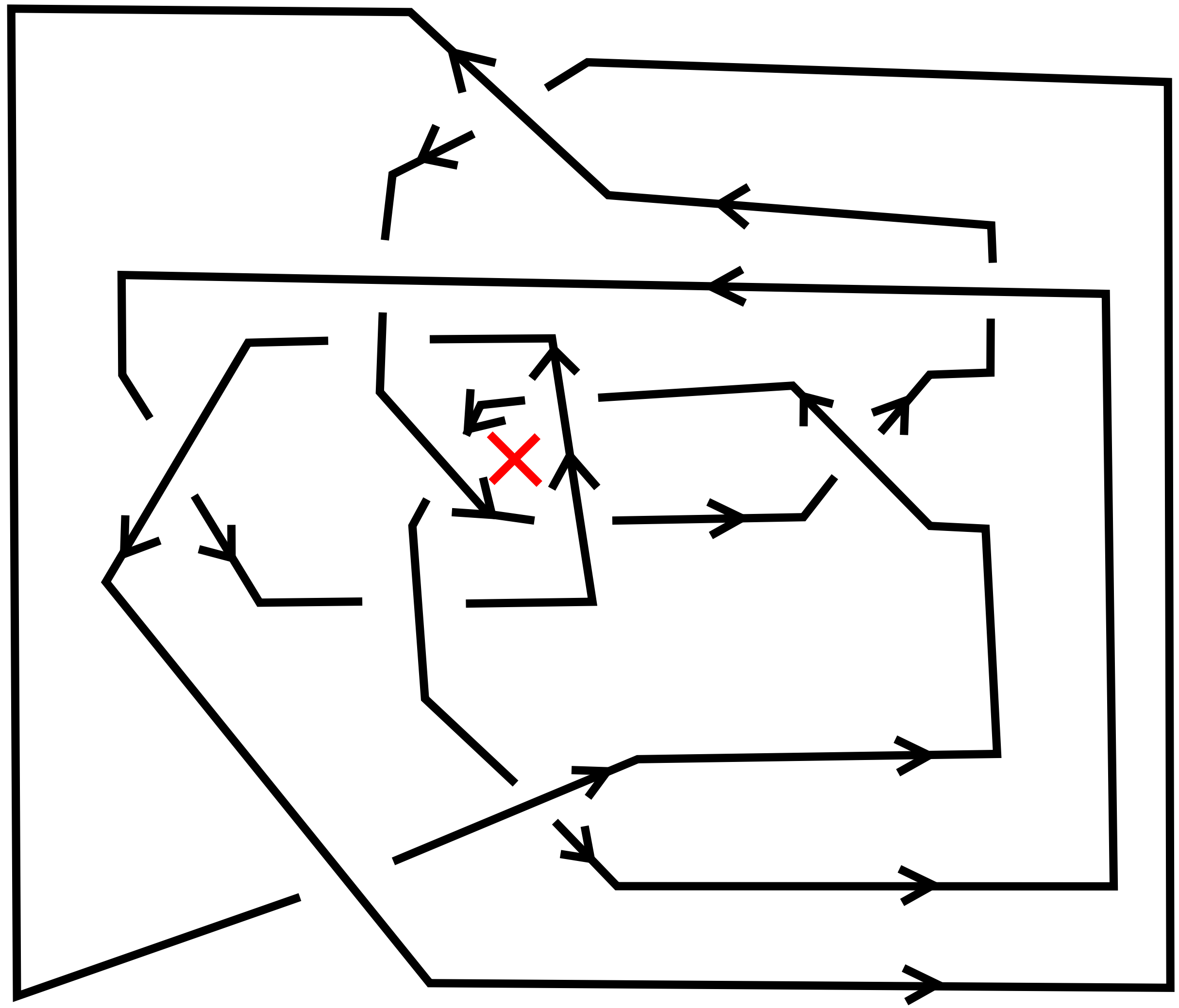
- **an upper bound** $(n = 1)$











Thank you.