Pseudo-fiber surface and unknotting operation for fibered links

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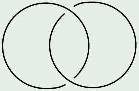
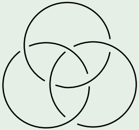
Definition 1.1

- *n-component link*: $S^1_1 \cup S^1_2 \cup \cdots \cup S^1_n \hookrightarrow \; S^3$
  mutually disjoint 1-spheres embedded in S^3
- *knot*: one component link

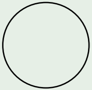
Example 1.2

link :

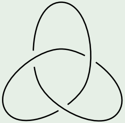




knot :



trivial knot



trefoil knot

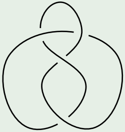
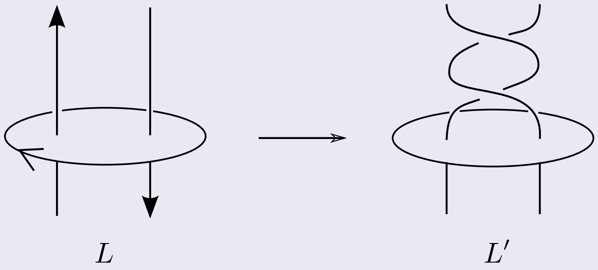


figure eight knot

Definition 1.3 (unknotting operation)



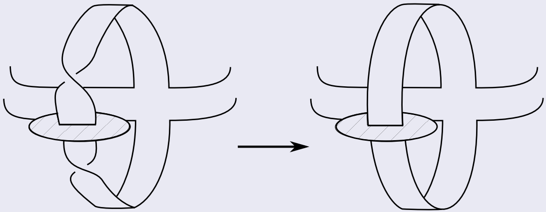
Definition 1.4

K :knot,
$u(K)$:=min{ the number of unknotting operations
required to obtain a trivial knot from K }
: *unknotting number* of K

Theorem 1.5 (Scharlemann-Thompson)

$L,\ L'$: knot
$L \to L'$: unknotting operation
$g(L') < g(L)$
$\exists S$: minimal genus Seifert surface of L
s.t. S is a plumbing of a surface and Hopf band,

and



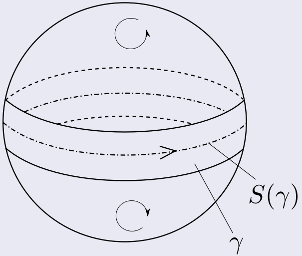
M. Sharlemann, A. Thomnpson, Link genus and Conway moves, Comment. Math. Helv. 64(1989), no.4, 527-53

Definition 1.6 (sutured manifold)

(M, γ)
is a *sutured manifold.*

$\underset{def}{\Longleftrightarrow}$ M : a 3−manifold

γ : union of annuli ($\subset \partial M$)



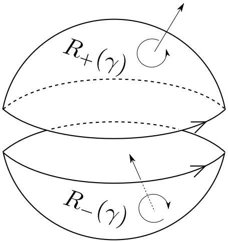
with $s(\gamma)$: oriented core curves of γ ,
where each component of $R(\gamma) := \partial M \setminus Int\gamma$ is oriented
so that the orientation meets $s(\gamma)$.

(Each component of $s(\gamma)$ is called a suture.)

Fix an orientation of M. Then:

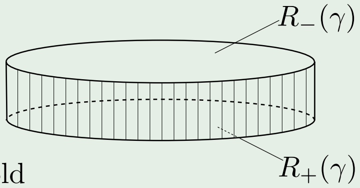
$R_+(\gamma)$($R_-(\gamma)$ resp.)

:=the union of the component of $R(\gamma)$
whose normal vector points out (into resp.) M.



Example 1.7 (product sutured manifold)

S: compact
orientable surface



$M = S \times [0,1]$
$\gamma = \partial S \times [0,1]$

Then (M, γ) is a sutured manifold
s.t. $R_+(\gamma) = S \times \{1\}$
$R_-(\gamma) = S \times \{0\}$

Such (M, γ) is called a product sutured manifold.

L : link
S : Seifert surface for L
$E(L)(:= S^3 \setminus IntN(L))$: exterior of L
(For simplicity we denote $S \cap E(L)$ by S.)

$N = N(S; E(L))$
$\delta = N \cap \partial E(L)$

The product sutured (N, δ) is called
the sutured manifold obtained from S.

Definition 1.8 (complementary sutured manifold)

$N^c = cl(E(L) \setminus N)$
$\delta^c = cl(\partial E(L) \setminus \delta)$
$R_\pm(\delta^c) = R_\mp(\delta)$

(N^c, δ^c) is called *the complementary sutured manifold* for S.

L : link

S : Seifert surface for L

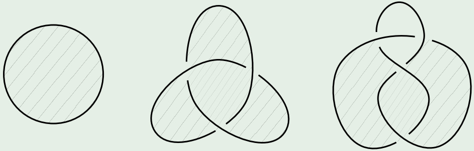
Definition 1.9 (fiber surface)

S is a *fiber surface* , if :
the complementary sutured manifold
is a product sutured manifold.

Definition 1.10 (fibered link)

L is a *fibered link* if
$\exists S$: Seifert surface for L s.t. S is a fiber surface.

Example 1.11



T. Kobayashi, Fiberd links and unknotting operations,
Osaka J. Math. 26(1989), 699-742

L : link
S : Seifert surface for L
(N^c, δ^c) : complementary sutured manifold of S

Definition 2.1 (pre-fiber surface)

S is a pre-fiber surface, if :

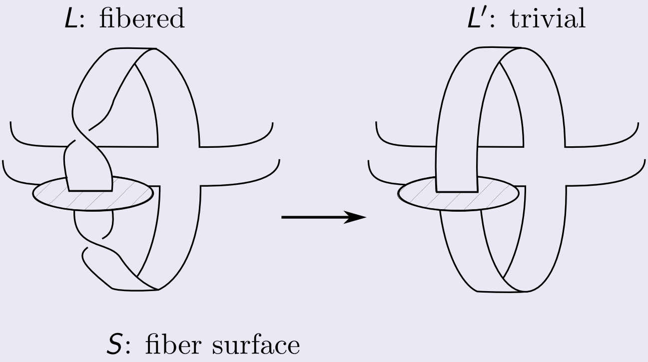
$\exists D^{\pm}(\subset N^c)$: mutually disjoint compressing disks of $R_{\pm}(\delta^c)$

s.t. $(N^{c\prime}, \delta^c)$ is a prodect sutured manifold,

where $N^{c\prime}$ is a 3-manifold obtained from N^c
by cutting along $D^+ \cup D^-$.

Theorem 2.2

Suppose L: fibered knot s.t. $u(L) = 1$.



Then S' is a pre-fiber surface.

Theorem 2.3

S:pre-fiber surface
with canonical compressing disks $\bar{D}^+$,$\bar{D}^-$.

Let α be an arc properly embedded in S.
s.t. $\alpha \cap \bar{D}^+$: 1-point,
$\alpha \cap \bar{D}^-$: 1-point.

The surface obtained from S
by applying a twist along α is a fiber surface.

Motivation:

Generalize the above results

for fibered knots with unknotting numbers > 1.

A generalization of pre-fiber surface.

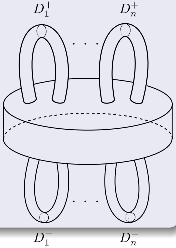
Let L: link
S: Seifert surface
(N^c, δ^c): complementary sutured manifold for S

Definition 3.1 (pseudo-fiber surface of level n)

For $n \geq 0$, S is a *pseudo-fiber surface of level n*, if:

$\exists D_1^{\pm}, \dots, D_n^{\pm}$
:mutually disjoint compressing disks of $R_{\pm}(\delta^c)$
s.t. $(N^{c\prime}, \delta^c)$is a product sutured manifold,
where $N^{c\prime}$ is a 3-manifold obtained from N^c
by cutting along

$$D_1^+ \cup \cdots \cup D_n^+ \cup D_1^- \cup \cdots \cup D_n^-$$



Particularly,

S is a pseudo-fiber surface of **level 0** $\Leftrightarrow$ S is a fiber surface

S is a pseudo-fiber surface of **level 1** $\Leftrightarrow$ S is a pre-fiber surface

A generalization of Theorem 2.6

Theorem 2.6

S:pre-fiber surface with canonical compressing disks $\bar{D}^+$,$\bar{D}^-$.
Let α be an arc properly embedded in S s.t. $\alpha \cap \bar{D}^+$: 1-point,
$\alpha \cap \bar{D}^-$: 1-point.
The surface obtained from S by applying a twist along α is a fiber surface.

Theorem 3.2

Let S: a pseudo-fiber surface of level n
$\alpha_1, \dots, \alpha_p$: mutually disjoint arcs properly embedded in S $(p \leq n)$
Suppose that $\exists \bar{D}_1^+, \dots, \bar{D}_n^+, \bar{D}_1^-, \dots, \bar{D}_n^-$
: a system of canonical compressing disks
s.t. $\alpha_i \cap \bar{D}_j^\pm = \emptyset$ *for* $\forall i, j$ $(i \neq j)$,
$\alpha_i \cap \partial \bar{D}_i^+$ $(\partial \bar{D}_i^-$ *resp.*$)$: 1-point. $(i = 1, \dots, p)$
Then
the surface obtained from S *by twisting along* $\alpha_1 \cup \cdots \cup \alpha_p$
is a pseudo-fiber surface of level $n - p$.

Definition 3.3 (Ascending sequence of pseudo-fiber surfaces)

Let S_i: pseudo-fiber surface of level p_i $(i = 0, 1, \ldots, n)$

A sequence of pseudo-fiber surfaces S_i of level p_i

$$S_0 \to S_1 \to \cdots \to S_i \to S_{i+1} \to \cdots \to S_n$$

$$\text{where } p_i \leq p_{i+1} \ (i = 0, 1, \ldots, n-1)$$

$$p_0 = 0$$

is an *ascending sequence of pseudo-fiber surface* if:

$\exists \alpha_1^{(i)}, \ldots, \alpha_{q_i}^{(i)}$: mutually disjoint arcs properly embedded in S_i
s.t. S_{i+1} is obtained from S_i by twisting along $\alpha_1^{(i)} \cup \cdots \cup \alpha_{q_i}^{(i)}$.

Question 3.4

For each fibered knot K,

does there exist an ascending sequence of pseudo-fiber surfaces realizing the unknotting number $u(K)$?

More precisely,
does there exist an ascending sequence of pseudo-fiber surfaces

$$S_0 \to S_1 \to \cdots \to S_n$$

(with levels p_i, arcs $\alpha_1^{(i)}, \ldots, \alpha_{q_i}^{(i)}$ as above)
such that $\partial S_0 = K$, and

$$u(K) = q_0 + q_1 + \cdots + q_{n-1}?$$

Torus knot

$T(p,q)$: torus knot of type (p,q), where $0 < p < q$

Theorem 3.5 (K-M)

$$u(T(p,q)) = \frac{(p-1)(q-1)}{2}$$

Consider the fractional expansion

$$\frac{q}{p} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{m-1} + \cfrac{1}{p_{m-1}}}}}, \quad (a_i > 0, p_{m-1} > 1).$$

Euclidian algorithm

$p_{i-1} = p_i a_{i+1} + p_{i+1}$ $(0 \le i \le m-2,\ p_{-1} = q,\ p_0 = p,\ 0 < p_{i+1} < p_i)$

Kronheimer, P. B.; Mrowka, T. S. , Gauge theory for embedded surfaces, I, Topology 32 (1993), no.4, 773−826

Theorem 3.6

$T(p,q)$: torus knot of type (p,q)
$\exists\ S_0 \to S_1 \to S_2 \to \cdots \to S_m$
: an ascending sequence of pseudo-fiber surfaces for $T(p,q)$ realizing the unknotting number of $T(p,q)$ with length m,
s.t. the level of S_i is $\sum_{k=0}^{i}(p_{k-1}-1)a_{k-1}$.

Example 3.7

Note that $u(T(5,7)) = 12$, Note $\dfrac{7}{5} = 1 + \dfrac{1}{2+\dfrac{1}{2}}$

$\exists\ S_0 \to S_1 \to S_2$: ascending sequence
realizing the unknotting number of $T(5,7)$,
where the level of S_1 is 4,
the level of S_2 is 6.