Epimorphisms between knot groups and the images of meridians

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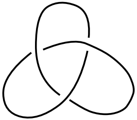
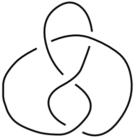
January 10, 2012

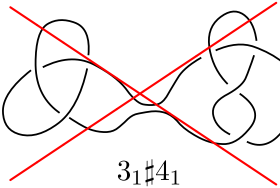
K : a *prime* knot in S^3



3_1



4_1



$3_1 \sharp 4_1$

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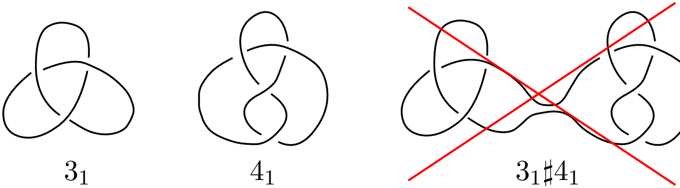

$* \in \partial(K \times D^2)$: a base point
$G(K)$: the knot group of K i.e. $G(K) = \pi_1(S^3 - K, *)$

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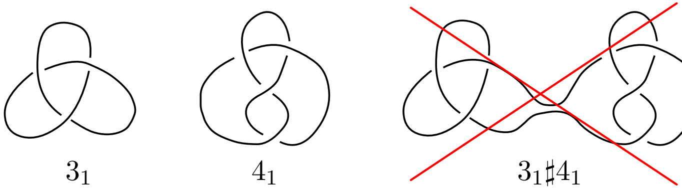

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$G(K)$: the knot group of K i.e. $G(K) = \pi_1(S^3 - K, *)$

$\mu_K \in G(K)$: meridian of K
i.e. $\mu_K = [\alpha_K]$ s.t. $\alpha_K \sim * \times \partial D^2$, $lk(K, \alpha_K) = 1$

Definition.

K, K' : two prime knots

$$K \geq K' \iff {}^{\exists}\varphi : \; G(K) \longrightarrow\!\!\!\!\rightarrow G(K')$$

$$K \geq_{\mu} K' \iff {}^{\exists}\varphi : \; \begin{array}{ccc} G(K) & \longrightarrow\!\!\!\!\rightarrow & G(K') \\ \mu_K & \longmapsto & \mu_{K'} \end{array}$$

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Fact.

The relation "$\geq$" is a partial order on the set of prime knots.

- $K \geq K$
- $K \geq K'$, $K' \geq K \implies K = K'$
- $K \geq K'$, $K' \geq K'' \implies K \geq K''$

The relation "$\geq_\mu$" is also a partial order

Theorem. (Kitano-S., Horie-Kitano-Matsumoto-S.)

The partial order "$\geq_\mu$" on the set of prime knots with up to 11 crossings is given by

$$\left.\begin{array}{l}
8_5, 8_{10}, 8_{15}, 8_{18}, 8_{19}, 8_{20}, 8_{21}, 9_1, 9_6, 9_{16}, 9_{23}, 9_{24}, 9_{28}, 9_{40}, \\
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11a_{338}, 11a_{340}, 11a_{351}, 11a_{352}, 11a_{355}, 11n_{71}, 11n_{72}, 11n_{73}, \\
11n_{74}, 11n_{75}, 11n_{76}, 11n_{77}, 11n_{78}, 11n_{81}, 11n_{85}, 11n_{86}, 11n_{87}, \\
11n_{94}, 11n_{104}, 11n_{105}, 11n_{106}, 11n_{107}, 11n_{136}, 11n_{164}, 11n_{183}, \\
11n_{184}, 11n_{185}
\end{array}\right\} \geq_\mu 3_1$$

$$\left.\begin{array}{l} 8_{18}, 9_{37}, 9_{40}, \\ 10_{58}, 10_{59}, 10_{60}, 10_{122}, 10_{136}, 10_{137}, 10_{138}, \\ 11a_{5}, 11a_{6}, 11a_{51}, 11a_{132}, 11a_{239}, 11a_{297}, 11a_{348}, 11a_{349}, \\ 11n_{100}, 11n_{148}, 11n_{157}, 11n_{165} \end{array}\right\} \geq_\mu 4_1$$

$$11n_{78}, 11n_{148} \geq_\mu 5_1$$

$$10_{74}, 10_{120}, 10_{122}, 11n_{71}, 11n_{185} \geq_\mu 5_2$$

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$$11a_{47}, 11a_{239} \geq_\mu 6_3$$

To determine the partial order $\geq_\mu$ on the set of prime knots

For each pair of two prime knots K, K',
determine whether there exists an epimorphism

$$\begin{array}{cccc} \varphi: & G(K) & \longrightarrow\!\!\!\!\rightarrow & G(K') \\ & \mu_K & \longmapsto & \mu_{K'} \end{array}$$

which preserves meridians.

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The number of prime knots with up to 11 crossings is $\mathbf{801}$.
Then the number of cases to consider is ${}_{801}\mathrm{P}_2 = \mathbf{640,800}$.

the existence of an epimorphism which preserves meridians

We can construct such an epimorphism explicitly

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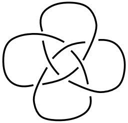
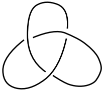
Example. $8_{18} \geq_\mu 3_1$?

$8_{18} =$  , $3_1 =$ 

$$G(8_{18}) = \left\langle \begin{array}{l} x_1, x_2, x_3, \\ x_4, x_5, x_6, \\ x_7, x_8 \end{array} \middle| \begin{array}{l} x_4 x_1 \bar{x}_4 \bar{x}_2, x_5 x_3 \bar{x}_5 \bar{x}_2, x_6 x_3 \bar{x}_6 \bar{x}_4, \\ x_7 x_5 \bar{x}_7 \bar{x}_4, x_8 x_5 \bar{x}_8 \bar{x}_6, x_1 x_7 \bar{x}_1 \bar{x}_6, \\ x_2 x_7 \bar{x}_2 \bar{x}_8 \end{array} \right\rangle$$

$$G(3_1) = \langle y_1, y_2, y_3 \,|\, y_3 y_1 \bar{y}_3 \bar{y}_2,\ y_1 y_2 \bar{y}_1 \bar{y}_3 \rangle$$

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$$\varphi : G(8_{18}) \longrightarrow G(3_1)$$

$$\begin{array}{llll} \varphi(x_1) = y_1, & \varphi(x_2) = y_2, & \varphi(x_3) = y_1, & \varphi(x_4) = y_3, \\ \varphi(x_5) = y_3, & \varphi(x_6) = y_1y_3\bar{y}_1, & \varphi(x_7) = y_3, & \varphi(x_8) = y_1 \end{array}$$

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$$\begin{array}{cccc} \varphi : & G(8_{18}) & \longrightarrow\!\!\!\!\rightarrow & G(3_1) \\ & \mu_{8_{11}} & \longmapsto & \mu_{3_1} \end{array}$$

$$8_{18} \geq_\mu 3_1$$

the non-existence of an epimorphism which preserves meridians

(1) By the (classical) Alexander polynomial

K : a knot

Δ_K : the Alexander polynomial of K

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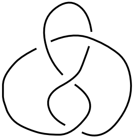
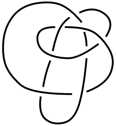
Fact.

K, K' : two knots
If Δ_K can not be divided by $\Delta_{K'}$,
$\Longrightarrow$ there exists no epimorphism $G(K) \longrightarrow\!\!\!\!\!\rightarrow G(K')$.

Example. $4_1 \geq_\mu 8_{21}$?

$4_1 =$  , $8_{21} =$ 

$\Delta_{4_1} = t^2 - 3t + 1, \qquad \Delta_{8_{21}} = t^4 - 4t^3 + 5t^2 - 4t + 1$

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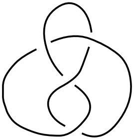
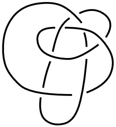
$4_1 =$  , $8_{21} =$ 

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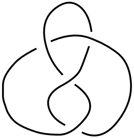
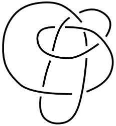
$4_1 =$  , $8_{21} =$ 

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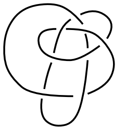
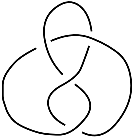
$8_{21} =$  , $4_1 =$ 

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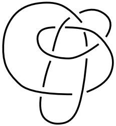
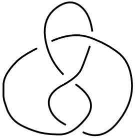
$8_{21} =$  , $4_1 =$ 

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Δ_{4_1} can divide $\Delta_{8_{21}}$!

We cannot determine the existence of an epimorphism from $G(8_{21})$ onto $G(4_1)$ by the Alexander polynomial.

the non-existence of an epimorphism which preserves meridians

(1) By the (classical) Alexander polynomial
(2) By the twisted Alexander polynomial

Fix Wirtinger presentation of $G(K)$
$\Delta_{K,\rho}$: the twisted Alexander polynomial of K
$\Delta^N_{K,\rho}, \Delta^D_{K,\rho}$: the numerator and denominator of $\Delta_{K,\rho}$

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Theorem. (Kitano-S.-Wada)

If there exists a representation $\rho' : G(K') \to SL(2; \mathbb{Z}/p\mathbb{Z})$ such that

$$\Delta^N_{K,\rho} \text{ is not divisible by } \Delta^N_{K',\rho'} \text{ or } \Delta^D_{K,\rho} \neq \Delta^D_{K',\rho'}$$

for any representation $\rho : G(K) \to SL(2; \mathbb{Z}/p\mathbb{Z})$,
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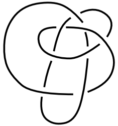
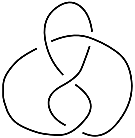
$8_{21} =$  , $4_1 =$ 

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For a certain representation $\rho' : G(4_1) \longrightarrow SL(2; \mathbb{Z}/3\mathbb{Z})$,

$$\Delta^N_{4_1,\rho'} = t^4 + t^2 + 1, \qquad \Delta^D_{4_1,\rho'} = t^2 + t + 1$$

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Table of the twisted Alexander polynomials of $G(8_{21})$ for all representations $\rho : G(8_{21}) \longrightarrow SL(2; \mathbb{Z}/3\mathbb{Z})$

	$\Delta^N_{8_{21},\rho_i}$	$\Delta^D_{8_{21},\rho_i}$
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ρ_2	$t^8 + t^7 + 2t^6 + 2t^4 + 2t^2 + t + 1$	$t^2 + t + 1$
ρ_3	$t^8 + t^7 + 2t^6 + 2t^4 + 2t^2 + t + 1$	$t^2 + 2t + 1$
ρ_4	$t^8 + 2t^7 + 2t^6 + 2t^4 + 2t^2 + 2t + 1$	$t^2 + t + 1$
ρ_5	$t^8 + 2t^7 + 2t^6 + 2t^4 + 2t^2 + 2t + 1$	$t^2 + 2t + 1$

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$\mathbf{146}$ cases: existence of an epimorphism
$\mathbf{637,501}$ cases : non-existence by the Alexander polynomial
$\mathbf{3,153}$ cases : non-existence by the twisted Alexander poly.

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11a_{212}, 11a_{216}, 11a_{223}, 11a_{231}, 11a_{232}, 11a_{236}, 11a_{244}, 11a_{245}, \\
11a_{261}, 11a_{263}, 11a_{264}, 11a_{286}, 11a_{305}, 11a_{306}, 11a_{318}, 11a_{332}, \\
11a_{338}, 11a_{340}, 11a_{351}, 11a_{352}, 11a_{355}, 11n_{71}, 11n_{72}, 11n_{73}, \\
11n_{74}, 11n_{75}, 11n_{76}, 11n_{77}, 11n_{78}, 11n_{81}, 11n_{85}, 11n_{86}, 11n_{87}, \\
11n_{94}, 11n_{104}, 11n_{105}, 11n_{106}, 11n_{107}, 11n_{136}, 11n_{164}, 11n_{183}, \\
11n_{184}, 11n_{185}
\end{array}\right\} \geq_\mu 3_1$$

$$\left.\begin{array}{l} 8_{18}, 9_{37}, 9_{40}, \\ 10_{58}, 10_{59}, 10_{60}, 10_{122}, 10_{136}, 10_{137}, 10_{138}, \\ 11a_{5}, 11a_{6}, 11a_{51}, 11a_{132}, 11a_{239}, 11a_{297}, 11a_{348}, 11a_{349}, \\ 11n_{100}, 11n_{148}, 11n_{157}, 11n_{165} \end{array}\right\} \geq_\mu 4_1$$

$$11n_{78}, 11n_{148} \geq_\mu 5_1$$

$$10_{74}, 10_{120}, 10_{122}, 11n_{71}, 11n_{185} \geq_\mu 5_2$$

$$11a_{352} \geq_\mu 6_1$$

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$$11a_{47}, 11a_{239} \geq_\mu 6_3$$

Definition.

K, K' : two prime knots

$$K \geq K' \iff {}^{\exists}\varphi : \; G(K) \longrightarrow\!\!\!\!\!\rightarrow G(K')$$

$$\begin{array}{rcl} K \geq_\mu K' \iff {}^{\exists}\varphi : & G(K) & \longrightarrow\!\!\!\!\!\rightarrow & G(K') \\ & \mu_K & \longmapsto & \mu_{K'} \end{array}$$

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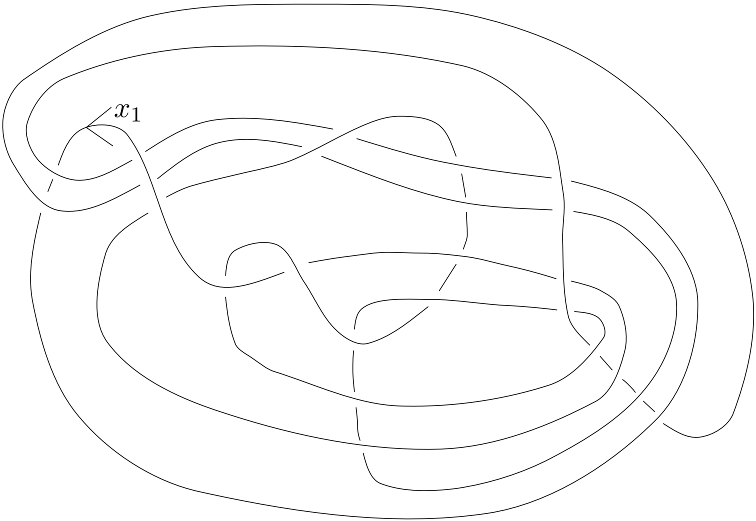
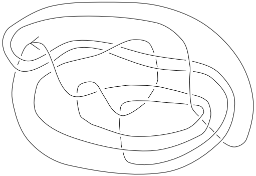
$K =$ 



Wirtinger presentation of $G(K)$
generators : $x_1, x_2, \ldots, x_{24}$
relators :

$$
\begin{array}{llll}
x_6x_2\bar{x}_6\bar{x}_1, & x_{10}x_2\bar{x}_{10}\bar{x}_3, & x_6x_3\bar{x}_6\bar{x}_4, & x_{22}x_4\bar{x}_{22}\bar{x}_5, \\
x_1x_6\bar{x}_1\bar{x}_5, & x_{17}x_7\bar{x}_{17}\bar{x}_6, & x_{23}x_7\bar{x}_{23}\bar{x}_8, & x_{13}x_9\bar{x}_{13}\bar{x}_8, \\
x_3x_9\bar{x}_3\bar{x}_{10}, & x_1x_{10}\bar{x}_1\bar{x}_{11}, & x_{22}x_{12}\bar{x}_{22}\bar{x}_{11}, & x_6x_{13}\bar{x}_6\bar{x}_{12}, \\
x_{23}x_{14}\bar{x}_{23}\bar{x}_{13}, & x_{17}x_{14}\bar{x}_{17}\bar{x}_{15}, & x_{18}x_{16}\bar{x}_{18}\bar{x}_{15}, & x_6x_{17}\bar{x}_6\bar{x}_{16}, \\
x_1x_{17}\bar{x}_1\bar{x}_{18}, & x_{16}x_{19}\bar{x}_{16}\bar{x}_{18}, & x_{24}x_{19}\bar{x}_{24}\bar{x}_{20}, & x_{12}x_{21}\bar{x}_{12}\bar{x}_{20}, \\
x_4x_{21}\bar{x}_4\bar{x}_{22}, & x_1x_{23}\bar{x}_1\bar{x}_{22}, & x_6x_{23}\bar{x}_6\bar{x}_{24}, & x_{18}x_{24}\bar{x}_{18}\bar{x}_1
\end{array}
$$

where $\bar{x}_i = {x_i}^{-1}$.

Wirtinger presentation of $G(3_1)$:

$$G(3_1) = \langle y_1, y_2 \,|\, y_1 y_2 y_1 = y_2 y_1 y_2 \rangle = \langle 1, 2 \,|\, 121 = 212 \rangle$$

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We define a map $f : G(K) \to G(3_1)$ as follows:

$$\begin{array}{ll}
f(x_1) = 12\bar{1}2\bar{1}, & f(x_2) = 1\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}2\bar{1}, \\
f(x_3) = 12\bar{1}2\bar{1}, & f(x_4) = 1\bar{2}12\bar{1}2\bar{1}21\bar{2}\bar{1}2\bar{1}, \\
f(x_5) = 212\bar{1}2\bar{1}21\bar{2}\bar{1}\bar{2}, & f(x_6) = 1\bar{2}12\bar{1}2\bar{1}2\bar{1}, \\
f(x_7) = 1\bar{2}1\bar{2}\bar{2}122\bar{1}\bar{1}222\bar{1}2\bar{1}, & f(x_8) = 1\bar{2}12\bar{1}2\bar{1}2\bar{1}, \\
f(x_9) = 1\bar{2}1\bar{2}\bar{2}12\bar{1}2\bar{1}22\bar{1}2\bar{1}, & f(x_{10}) = 1\bar{2}12\bar{1}2\bar{1}2\bar{1}, \\
f(x_{11}) = 122\bar{1}\bar{1}, & f(x_{12}) = 1\bar{2}\bar{2}122\bar{1}\bar{1}22\bar{1}, \\
f(x_{13}) = 12\bar{1}2\bar{1}, & f(x_{14}) = 1\bar{2}1\bar{2}\bar{2}\bar{2}12\bar{1}2\bar{1}222\bar{1}2\bar{1}, \\
f(x_{15}) = 12\bar{1}2\bar{1}, & f(x_{16}) = 1\bar{2}\bar{2}12\bar{1}2\bar{1}22\bar{1}, \\
f(x_{17}) = 1\bar{2}1\bar{2}1\bar{2}\bar{1}\bar{2}12\bar{1}2\bar{1}212\bar{1}2\bar{1}2\bar{1}, & f(x_{18}) = 22\bar{1}, \\
f(x_{19}) = 1\bar{2}\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}22\bar{1}, & f(x_{20}) = 22\bar{1}, \\
f(x_{21}) = 1\bar{2}12\bar{1}\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1}212\bar{1}2\bar{1}21\bar{2}\bar{1}2\bar{1}, & f(x_{22}) = 22\bar{1}, \\
f(x_{23}) = 1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}, & f(x_{24}) = 1\bar{2}12\bar{1}\bar{1}221\bar{2}\bar{1}2\bar{1}.
\end{array}$$

Theorem.

The above mapping $f : G(K) \longrightarrow G(3_1)$ is an epimorphism which does not map a meridian of K to a meridian of 3_1. Moreover, there does not exist an epimorphism from $G(K)$ onto $G(3_1)$ which preserves meridians.

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Corollary.

$$K \geq K' \not\Rightarrow K \geq_\mu K'$$

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\begin{aligned}
& f(x_6 x_2 \bar{x}_6 \bar{x}_1) \\
& \quad = \quad 1\bar{2}12\bar{1}2\bar{1}2\bar{1} \cdot 1\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}2\bar{1} \cdot 1\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1} \cdot 1\bar{2}1\bar{2}\bar{1} \\
& \quad = \quad e \\
& f(x_{10} x_2 \bar{x}_{10} \bar{x}_3) \\
& \quad = \quad 1\bar{2}12\bar{1}2\bar{1}2\bar{1} \cdot 1\bar{2}1\bar{2}1\bar{2}\bar{1}222\bar{1}2\bar{1}2\bar{1} \cdot 1\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1} \cdot 1\bar{2}1\bar{2}\bar{1} \\
& \quad = \quad e \\
& f(x_6 x_3 \bar{x}_6 \bar{x}_4) \\
& \quad = \quad 1\bar{2}12\bar{1}2\bar{1}2\bar{1} \cdot 12\bar{1}2\bar{1} \cdot 1\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1} \cdot 1\bar{2}12\bar{1}\bar{2}1\bar{2}1\bar{2}\bar{1}2\bar{1} \\
& \quad = \quad e \\
& \qquad \ldots
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&\quad = \; f(x_{18})f(x_6)\overline{f(x_1)}\,\overline{f(x_1)}f(x_{18})f(x_6)\overline{f(x_1)} \\
&\quad = \; 22\bar{1}\cdot 1\bar{2}12\bar{1}2\bar{1}2\bar{1}\cdot 1\bar{2}12\bar{1}\cdot 1\bar{2}12\bar{1}\cdot 22\bar{1}\cdot 1\bar{2}12\bar{1}2\bar{1}2\bar{1}\cdot 1\bar{2}12\bar{1} \\
&\quad = \; 212\bar{1}\bar{2}1\bar{2}\bar{1}212\bar{1}\bar{1} = 121\bar{1}\bar{2}1\bar{2}\bar{1}121\bar{1}\bar{1} = 1, \\
& f(x_1\bar{x}_6\bar{x}_{18}x_1x_{18}x_6\bar{x}_1\bar{x}_1x_{18}x_6\bar{x}_1) \\
&\quad = \; \cdots \\
&\quad = \; 2
\end{aligned}
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Fix Wirtinger representations of the knot groups.

$\Delta_{K,\rho}$: the twisted Alexander polynomial of K

$\Delta^N_{K,\rho}, \Delta^D_{K,\rho}$: the numerator and denominator of $\Delta_{K,\rho}$

Theorem. (Kitano-S.-Wada)

If there exists a representation $\rho' : G(K') \to SL(2; \mathbb{Z}/p\mathbb{Z})$ such that

$$\Delta^N_{K,\rho} \text{ is not divisible by } \Delta^N_{K',\rho'} \text{ or } \Delta^D_{K,\rho} \neq \Delta^D_{K',\rho'}$$

for any representation $\rho : G(K) \to SL(2; \mathbb{Z}/p\mathbb{Z})$,
then there exists no epimorphism $G(K) \twoheadrightarrow G(K')$ which preseves meridians.

For a certain representation $\rho' : G(3_1) \longrightarrow SL(2;\mathbb{Z}/5\mathbb{Z})$,

$$\Delta^N_{3_1,\rho'} = t^4 + 2t^3 + 2t^2 + 2t + 1, \qquad \Delta^D_{3_1,\rho'} = t^2 + 2t + 1$$

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Table of the twisted Alexander polynomials of $G(K)$
for all representations $\rho : G(K) \longrightarrow SL(2; \mathbb{Z}/5\mathbb{Z})$

	Δ^N_{K,ρ_i}	Δ^D_{K,ρ_i}
ρ_1	$t^8 + 3t^6 + 3t^4 + 3t^2 + 1$	$t^2 + 1$
ρ_2	$t^8 + 4t^6 + t^4 + 4t^2 + 1$	$t^2 + 1$
ρ_3	$t^8 + t^7 + 4t^5 + 4t^4 + 4t^3 + t + 1$	$t^2 + 3t + 1$
ρ_4	$t^8 + 2t^7 + t^5 + 3t^4 + t^3 + 2t + 1$	$t^2 + t + 1$
ρ_5	$t^8 + 2t^7 + t^6 + 2t^5 + 4t^4 + 2t^3 + t^2 + 2t + 1$	$t^2 + t + 1$
ρ_6	$t^8 + 3t^7 + 4t^5 + 3t^4 + 4t^3 + 3t + 1$	$t^2 + 4t + 1$
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$$K \not\geq_\mu 3_1$$

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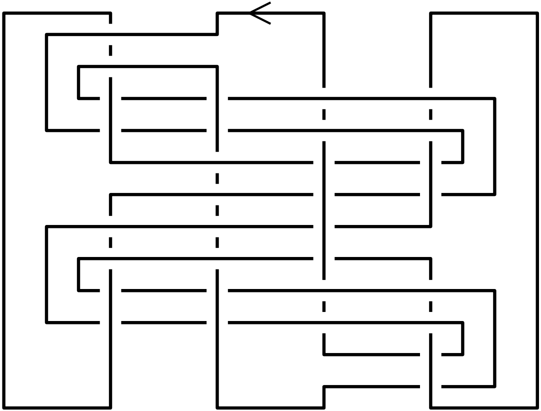
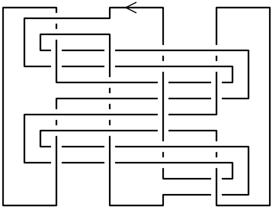


$K' =$ 

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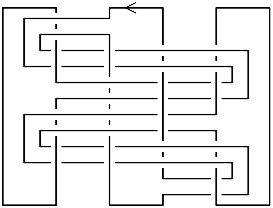
$K' =$ 

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Problem.

$$K' \geq_\mu 4_1 \ ?$$

Does there exist an epimorphism from $G(K')$ onto $G(4_1)$ which preserves meridians?

$$K \geq_\mu K' \Longrightarrow K \geq K', \qquad K \geq K' \not\Longrightarrow K \geq_\mu K'$$

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Determine the relation $\geq$ on the set of prime knots with up to 10 (or 11) crossings.

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- $8_{11} \not\geq_\mu 3_1$ by twisted Alexander polynomial
- $8_{11} \geq 3_1$?

Observation.

$$K \geq K' \Longleftrightarrow K \geq_\mu K'$$

with up to 9 crossings exept for a pair $(8_{11}, 3_1)$

Problem.

$$8_{11} \geq 3_1?$$

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Theorem. (Kitano-S.-Wada)

If there exists a representation $\rho' : G(K') \to SL(2; \mathbb{Z}/p\mathbb{Z})$ such that

$\Delta_{K,\rho}$ is not divisible by $\Delta_{K',\rho'}$

for any representation $\rho : G(K) \to SL(2; \mathbb{Z}/p\mathbb{Z})$,
then there exists no epimorphism $G(K) \twoheadrightarrow G(K')$.

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- We cannot find $\rho' : G(3_1) \to SL(2; \mathbb{Z}/p\mathbb{Z})$ for a prime number $p \leq 47$ such that the above theorem works for a pair $(8_{11}, 3_1)$.

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- We cannot find $\rho' : G(3_1) \to SL(2;\mathbb{Z}/p\mathbb{Z})$ for a prime number $p \leq 47$ such that the above theorem works for a pair $(8_{11}, 3_1)$.
- We cannot find an epimorphism $G(8_{11}) \longrightarrow\!\!\!\!\!\!\rightarrow G(3_1)$ yet.

Theorem. (Kitano-S.-Wada)

If there exists a representation $\rho' : G(K') \to SL(2; \mathbb{Z}/p\mathbb{Z})$ such that

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If there exists a representation $\rho' : G(K') \to SL(2; \mathbb{Z}/p\mathbb{Z})$ such that

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for any representation $\rho : G(K) \to SL(2; \mathbb{Z}/p\mathbb{Z})$,
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Problem.

Is the converse true?
If there exists no epimorphism $G(K) \longrightarrow\!\!\!\!\!\rightarrow G(K')$, then does there exist a prime number p and $\rho' : G(K') \to SL(2; \mathbb{Z}/p\mathbb{Z})$ such that

$$\Delta_{K,\rho} \text{ is not divisible by } \Delta_{K',\rho'}$$

for any $\rho : G(K) \to SL(2; \mathbb{Z}/p\mathbb{Z})$?