On a topological interpretation of quandle cocycle invariants of classical links

野坂 武史 (Takefumi Nosaka)

Kyoto University

Question & History

Q. Give a t.p.l. meaning of a quandle cocycle invariants.

X: a finite quandle

$L \subset S^3$: link

(Fenn-Rourke-Sanderson, 1996*)

Quandle homotopy invariant

$$\Xi_X(L) \in \mathbb{Z}[\,\pi_2(BX)\,]$$

quandle cocycle invariant	generalized cocycle invariant	shadow cocycle invariant	symmetric cocycle invariant
(Carter -Jelsovsky -Kamada -Langford -Saito, 1999*)	(Carter -Elhamdadi -Graña -Saito, 2004)	(Carter -Kamada -Saito, 2001)	(Kamada -Oshiro, 2010)

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Thm. (N.)

For "some" quandles X,

take the 1-st stage of the Postnikov tower:

$$H_3(\pi_1(BX)) \to \pi_2(BX) \xrightarrow{\mathcal{H}} H_2(BX) \to H_2(\pi_1(BX)) \to 0$$

(exact)

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$$\begin{tikzcd} H_3(\pi_1(BX)) \arrow[r] & \pi_2(BX) \arrow[r, "\mathcal{H}"] \arrow[l, bend left, "\exists"] & H_2(BX) \arrow[r] & H_2(\pi_1(BX)) \arrow[r] & 0 \end{tikzcd}$$

(exact)

is a split.

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Fact. (Eisemann) $\mathcal{H}(\Xi_X(L)) =$ "Colouring polynomial" [Eis]

"**Cor.**"

$$\begin{pmatrix} \text{q'dl homotopy} \\ \text{inv. } \Xi_X(L) \end{pmatrix} = (\text{Colouring poly.}) + \begin{pmatrix} \text{"Dijkgraaf-Witten"} \\ \text{inv. } \mathrm{DW}_\phi(\widehat{C}_L^\ell) \end{pmatrix}$$

$\widehat{C}_L^\ell$: cyclic ℓ-fold cov. branched over L.

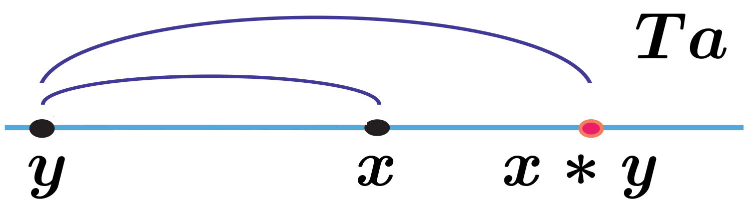
Goal: Explanation of the split.

Contents

Def. A quandle is $\begin{cases} Q & : \text{a set} \\ * & : Q \times Q \longrightarrow Q \end{cases}$ s.t.

- $^\forall x \in Q, \qquad x * x = x$
- $^\forall x, y \in Q, \qquad x = {}^{\exists !} z * y$
- $^\forall x, y, z \in Q, \qquad (x * y) * z = (x * z) * (y * z)$

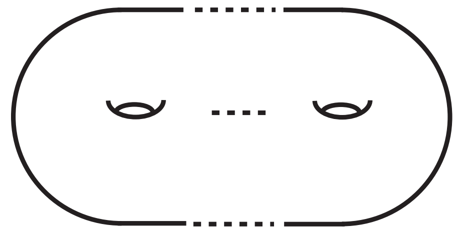
Ex. Alexander quandle on a finite field $X = \mathbb{F}_q$



$$x * y \overset{\text{def}}{=} y + \omega(x - y)$$

$(\bullet * y) \doteq$ 'ω multiple centered at y'

Ex. Symplectic quandle



Σ_g: ori. closed surface of genus g

$X = H_1(\Sigma_g; \mathbb{F}_q)$

$\langle\ ,\ \rangle : H_1(\Sigma_g; \mathbb{F}_q) \otimes H_1(\Sigma_g; \mathbb{F}_q) \to \mathbb{F}_q$: symplectic form.

$$x * y := \langle x, y \rangle \cdot y + x$$

Def. (Associated group)

Let X be a quandle

$$\mathrm{As}(X) := \langle\, x \in X \mid y \cdot x = (x * y) \cdot y \quad (\forall x, y \in X)\, \rangle$$

Def. X is of type $\ell \overset{\text{def}}{\Longleftrightarrow} \ell = \min\{n \mid (x \overbrace{* y) \cdots * y}^{n\text{-times}} = x\}$.

Ex. X: a symplectic q'dl $\Longrightarrow$ $\mathrm{type}(X) = \mathrm{char}(\mathbb{F}_q) = p$.

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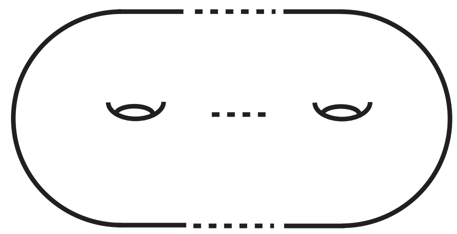

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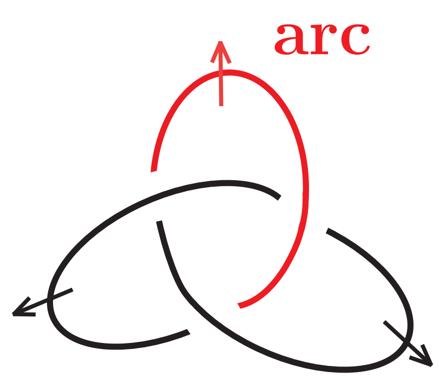
Ex. X: a symplectic q'dl $\implies$ $\mathrm{type}(X) = \mathrm{char}(\mathbb{F}_q) = p$.

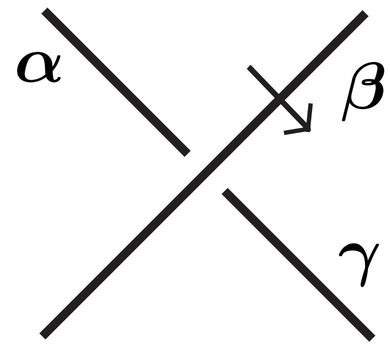
$(g, q) \neq (1, 3)$ or $p \neq 2 \implies \mathrm{As}(X) = \mathbb{Z} \times Sp(2g; \mathbb{F}_q)$.

Def. X : a quandle

D :an oriented link-diagram of L.



An X-coloring is a map $C : \{ \text{ arcs of } D \} \to X$ s.t.



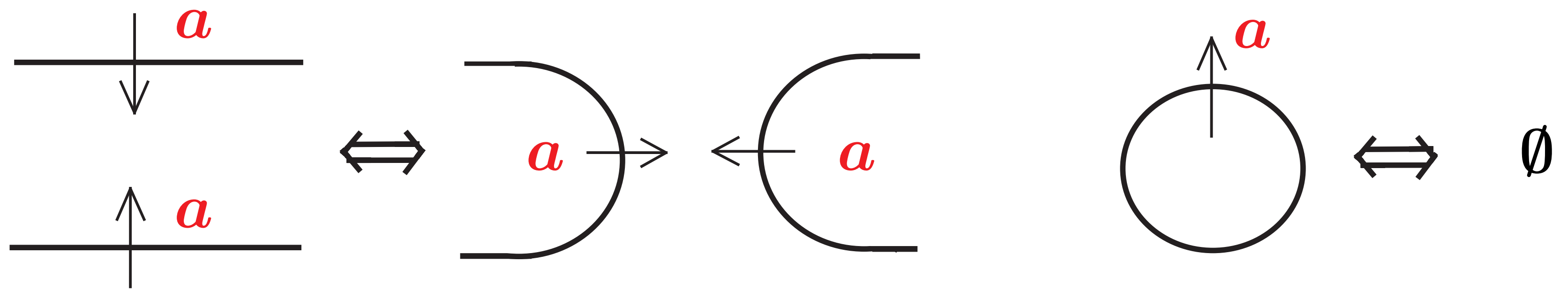
$$C(\gamma) = C(\alpha) * C(\beta)$$

Notation $\mathrm{Col}_X(D) := \{ X\text{-coloring of } D \}$

Property $D \overset{\text{Rmove}}{\longleftrightarrow} D' \implies \mathrm{Col}_X(D) \overset{\exists 1:1}{\longleftrightarrow} \mathrm{Col}_X(D')$

$\therefore$ X is finite $\implies \#(\mathrm{Col}_X(D))$ is a link inv.

$\overline{\Pi}(X) \overset{\text{def}}{=} \{X\text{-coloring of } D\}_D /$ R-moves, concordance rel.



- $\Pi(X)$ is an Abel grp. by $C_1 + C_2 := C_1 \sqcup C_2$.

Def. [Fenn-Rourke-Sanderson]

Let $|X| < \infty$. Quandle homotopy inv. is

$$\overline{\Xi_X}(L) := \sum_{C \in \mathrm{Col}_X(D)} [C] \in \mathbb{Z}[\Pi(X)].$$

Rem. $\begin{pmatrix} \forall\text{“quandle cocycle} \\ \text{inv.”of links.} \end{pmatrix} \longleftarrow \begin{pmatrix} \text{quandle homotopy} \\ \text{inv.} \in \mathbb{Z}[\Pi(X)] \end{pmatrix}$

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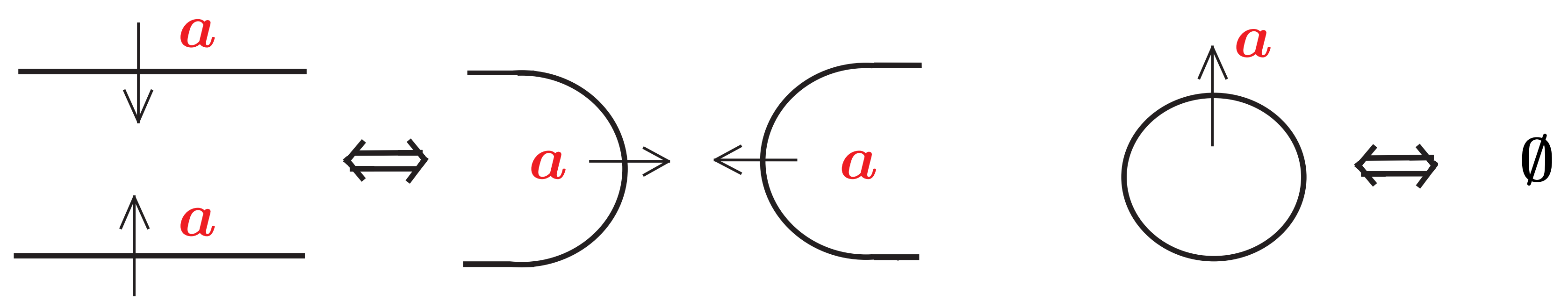

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Let's study the universal obj.

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The group $\Pi(X)$ VS $\pi_2(BX)$

Fenn-Rourke-Sanderson defined a ”classifying space” BX.

Properties.[FRS] Let X be a quandle

- $\pi_2(BX) \cong \Pi(X)$
- $\pi_1(BX) \cong \mathrm{As}(X) = \langle\, x \in X \mid y \cdot x = (x * y) \cdot y \,\rangle$.
- The action $\pi_1(BX) \curvearrowright \pi_2(BX)$ is trivial.

Classical way to compute π_2 (Cartan, Serre)

Take the 1-st stage of the Postnikov tower:

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$\exists$ (exact)

I now construct it topologically.

§3 Construction of the split

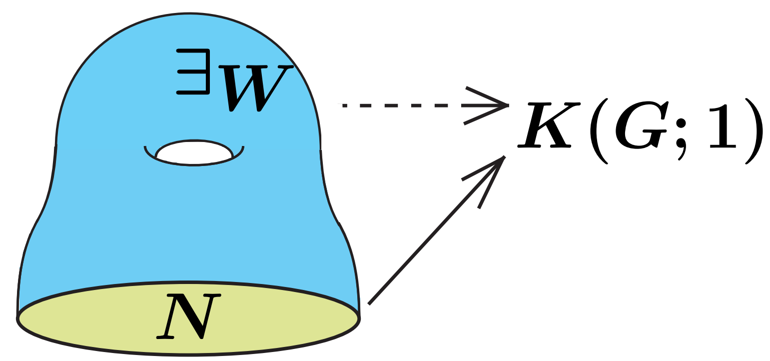
Review of G-bordism grp. [Thom] G: a group.

$$\Omega_n(G) := \left\{ (N, \pi_1(N) \xrightarrow{f} G) \mid N : \text{cl.}n\text{-mfd} \right\} / G\text{-cobordant}.$$

Here $(N, f : \pi_1(N) \to G)$ is G-cobordant.

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists W$: (n + 1)-mfd s.t. $\partial W = N$

$$\begin{tikzcd}
\pi_1(W) \arrow[r, dashed, "\exists \bar{f}"] & G \\
\pi_1(N) \arrow[u, "i_*"] \arrow[ur, "f"'] &
\end{tikzcd}$$



Rem $(n = 3)$

$$\Omega_3(G) \cong H_3(G; \mathbb{Z}) \cong H_3(K(G, 1); \mathbb{Z}).$$

From $\Pi(X)$ to $H_3(\mathrm{As}(X))$.

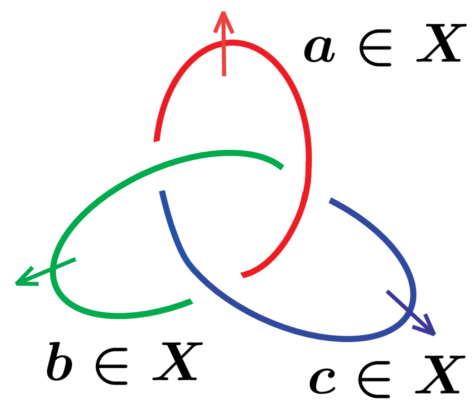
Let X be a quandle of type ℓ.

Set an epi. $\mathrm{As}(X) = \langle\, x \in X | y \cdot x = (x * y) \cdot y \,\rangle \xrightarrow{\epsilon} \mathbb{Z} \quad (x \mapsto 1)$

Given an X-coloring C of $L \subset S^3$

$\Downarrow$ Wirtinger presentation

$$\pi_1(S^3 \setminus L) \longrightarrow \mathrm{As}(X) \quad \text{grp. homo.}$$



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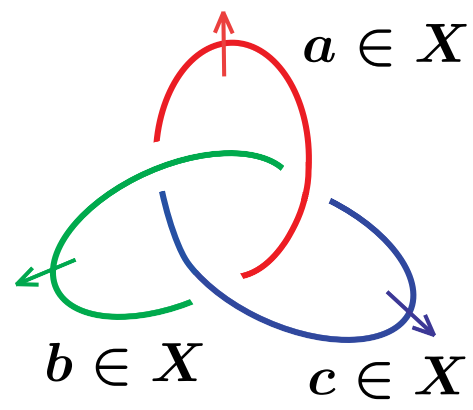
$$\begin{tikzcd}
\mathbb{Z} \arrow[r, equal] & \mathbb{Z} \\
\pi_1(S^3 \setminus L) \arrow[u, two heads] \arrow[r] & \mathrm{As}(X) \arrow[u, two heads] & \text{grp. homo.} \\
\pi_1(\widetilde{S^3 \setminus L}) \arrow[u, phantom, "\cup"] \arrow[r] \arrow[dr] & \mathrm{Ker}(\epsilon) \arrow[u, phantom, "\cup"] \\
 & \pi_1(\widehat{C}_L^\ell)
\end{tikzcd}$$



$\widetilde{S^3 \setminus L}$: abelian covering of L

$\widehat{C}_L^\ell$: cyclic ℓ-fold cov. branched over L

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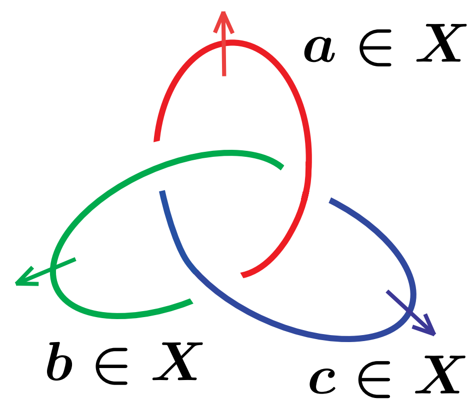

$\widetilde{S^3 \setminus L}$: abelian covering of L

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In summary, we get a map

$$\{\text{C: } X\text{-coloring of } D\ \} \longrightarrow \mathrm{Hom}_{\mathrm{gr}}(\pi_1(\widehat{C}^{\ell}_{L}),\ \mathrm{As}(X))$$

In summary, consider all the link-diagrams:

$$\Big\{ C : X\text{-coloring of } D \Big\}_D \longrightarrow \Big\{ \mathrm{Hom}(\pi_1(\widehat{C}^{\ell}_L),\ \mathrm{As}(X)\ ) \Big\}_L$$

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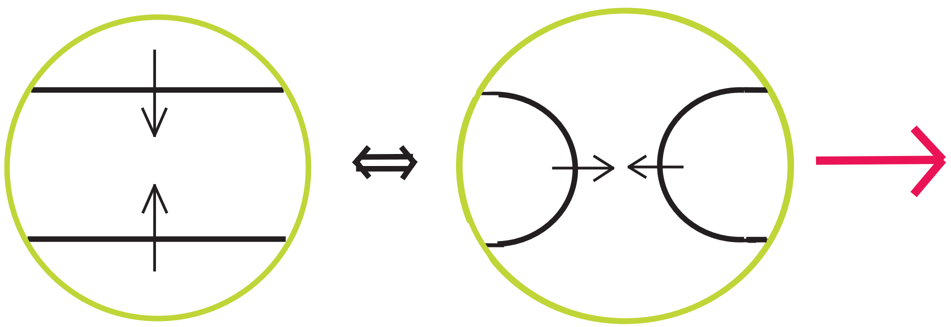
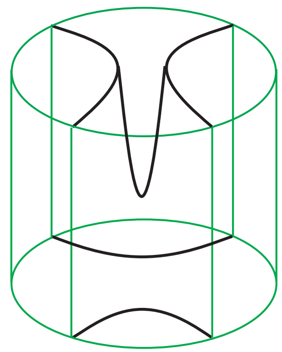
$\cup$ $\qquad\qquad\qquad\qquad$ $\cup$



ℓ-fold branched covering of



Prop. (N.)

This induces a hom. $\Pi(X) \to \Omega_3(\mathrm{As}(X)) = H_3(\pi_1(BX))$.

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For "some" quandles X (e.g. Alenxander or symplectic q'dl)

$$H_3(\pi_1(BX)) \to \pi_2(BX) \xrightarrow{\mathcal{H}} H_2(BX) \to H_2(\pi_1(BX)) \to 0$$

(exact)

The homomorphim gives its split.

Ex. $(X = H_1(\Sigma_g; \mathbb{F}_q)$: symplectic q'dl with $g > 5)$

Recall $\pi_1(BX) = \mathbb{Z} \times Sp(2g; \mathbb{F}_q)$.

Fact D. Quillen calculated $H_*(Sp(2g; \mathbb{F}_q)$.

Then the sequence becomes

$$\mathbb{Z}/(q^2 - 1) \to \pi_2(BX) \xrightarrow{\mathcal{H}} \mathbb{Z} \oplus (\mathbb{Z}/p)^h \to 0 \qquad (q = p^h)$$

$$\therefore \Pi(X) = \pi_2(BX) = \mathbb{Z}/(q^2 - 1) \oplus (\mathbb{Z}/p)^h.$$

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$\therefore\ \Pi(X) = \pi_2(BX) = \mathbb{Z}/(q^2-1) \oplus (\mathbb{Z}/p)^h$.

Thank you

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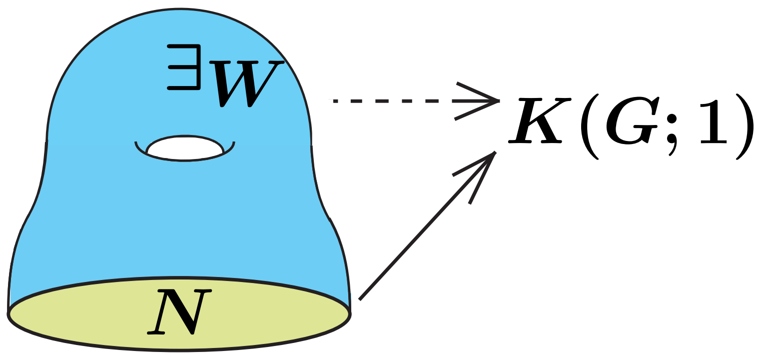

Def. G: a finite group. M: n-mfd.

Bordism D-W inv. is

$$\sum_{f \in \mathrm{Hom}(\pi_1(M), G)} [(M, \pi_1(M) \xrightarrow{f} G)] \in \mathbb{Z}[\Omega_n(G,c)],$$