Meridional destabilizing number and connected sums of knots

Toshio Saito

(Joetsu University of Education)

Standard positions of knots

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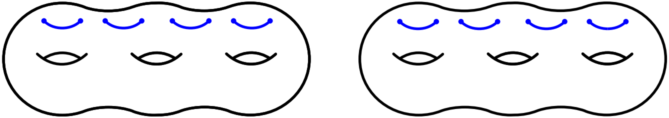
(1) Bridge position



If we take the Heegaard sphere of S^3, this is a classical bridge position.

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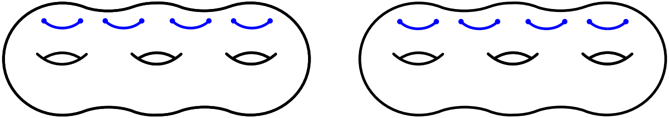

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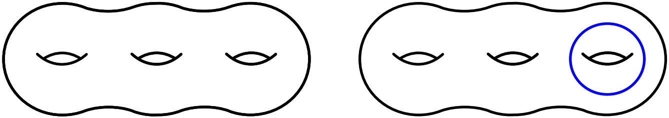
(2) Core position



This corresponds to *tunnel number*:

$\{$the minimal genus of a Heegaard surface giving a core position$\} - 1$.

Meridional destabilizing number

$K \subset S^3$: a 2-bridge knot

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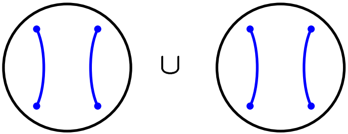
$(S^3, K) =$ 

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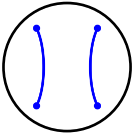
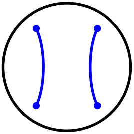
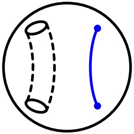
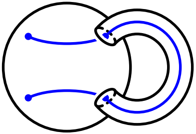
$(S^3, K) =$  $\cup$ 

$=$  $\cup$ 

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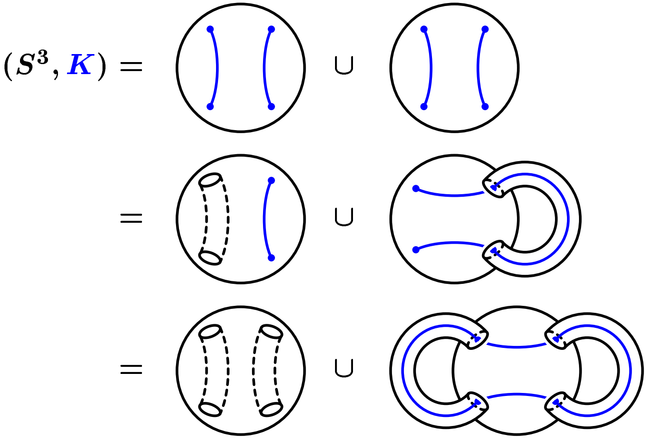

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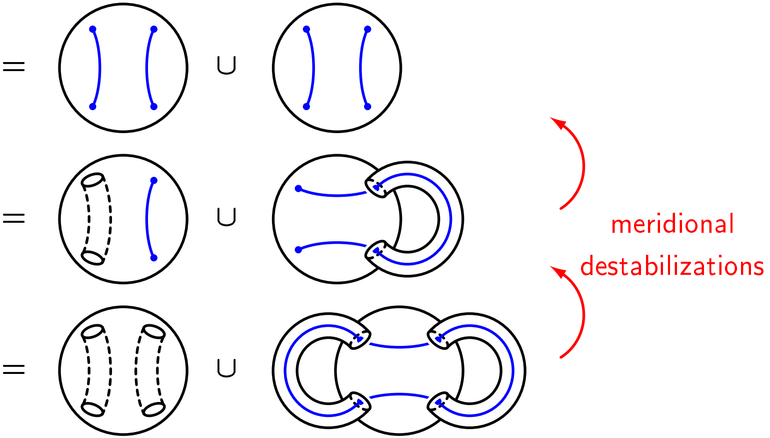
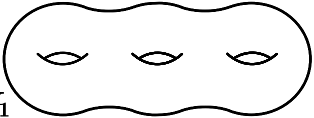
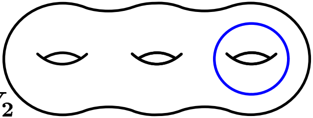
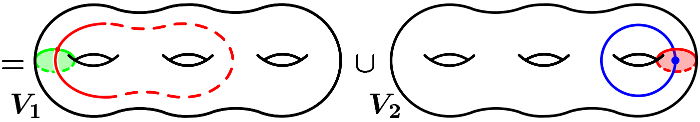


Meridional destabilizing number

$(M, K) =$  $\cup$ 

V_1 V_2

Meridional destabilizing number

$(M, K) =$ 

Definition.

A Heegaard splitting $V_1 \cup V_2$ of (M, K) is *meridionally stabilized*
$\overset{\text{def}}{\Longleftrightarrow}$ $\exists D_i$ $(i = 1, 2)$: meridian disks of V_i s.t. $\partial D_1 \cap \partial D_2 = \{1\text{pt}\}$ and $D_2 \cap K = \{1\text{pt}\}$.

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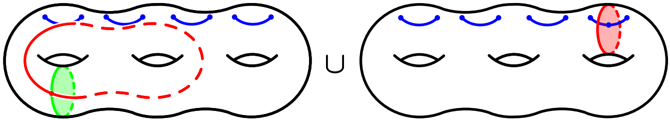
$(M, K) =$ 

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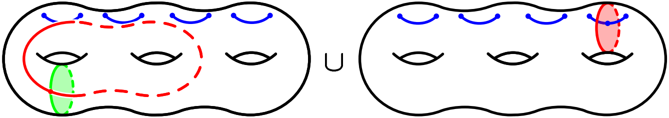
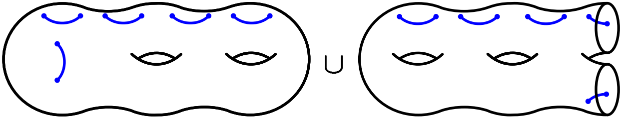


a (g, b)-splitting of (M, K)

a meridional destabilization



a $(g-1, b+1)$-splitting of (M, K)

We can make a Heegaard surface "smaller"

as K is in a *standard* position.

Definition.

Meridional destabilizing number of $K \subset M$ is defined by

$$md(K) = \max \left\{ n \;\middle|\; \begin{array}{c} \text{A minimal genus Heegaard splitting} \\ \text{of } (M, K) \text{ admits} \\ n \text{ times meridional destabilizations.} \end{array} \right\}$$

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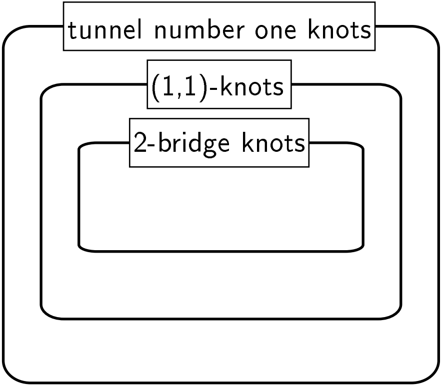
Example.

If K is a (non-trivial) 2-bridge knot, then $md(K) = 2$.

Meridional destabilizing number



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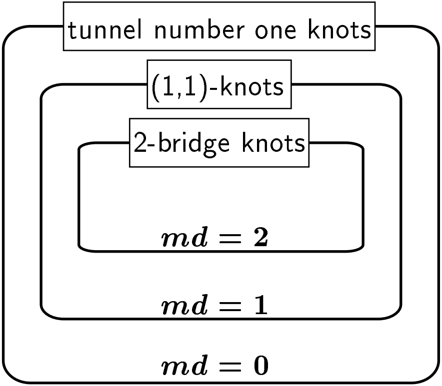

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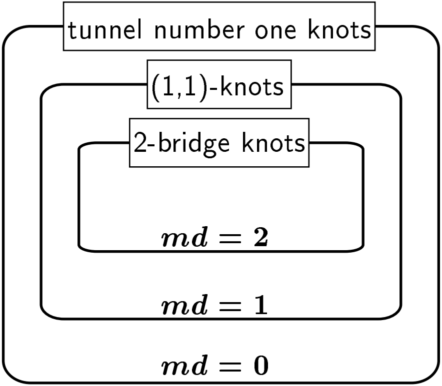

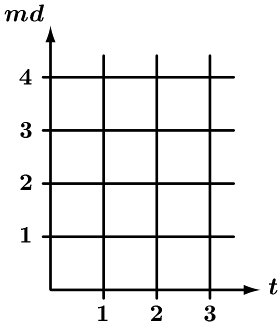

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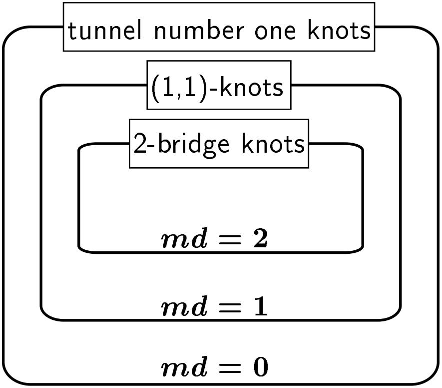

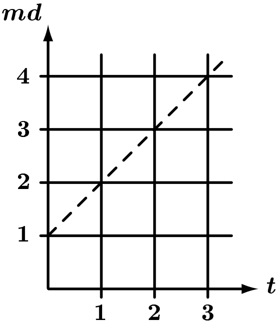

The connected sum of tunnel number one knots

$nK = K\#\cdots\#K$: the connected sum of n copies of $K \subset S^3$.

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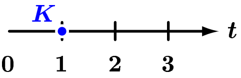

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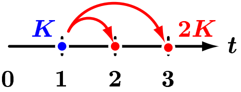

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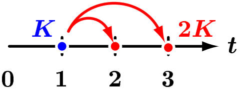

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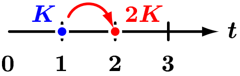
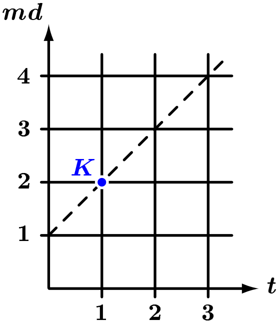
(2) (Morimoto)



$\Longleftrightarrow \quad md(K) \geq 1$.

2-bridge knots



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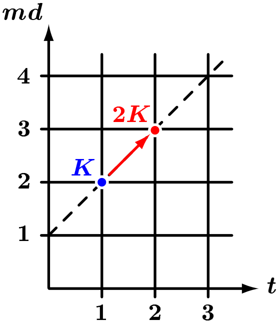

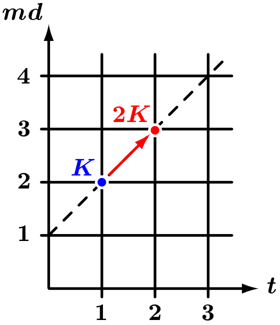

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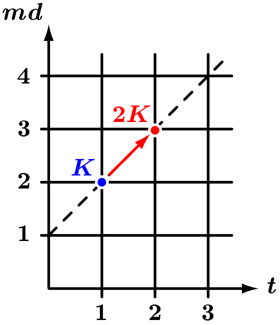

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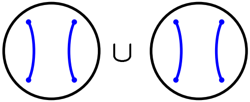
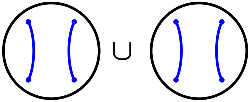
- $K =$ 

  $K =$ 

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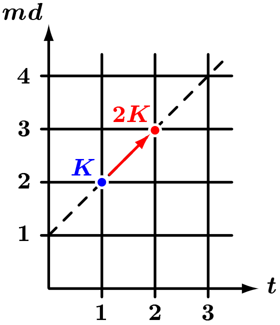

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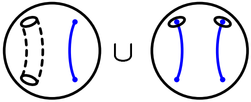
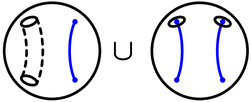
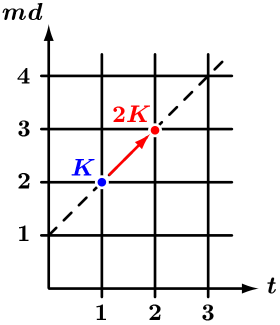
- $K =$ 

  $K =$ 



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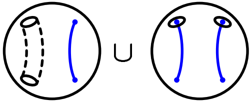
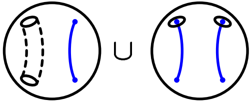
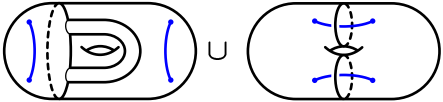
- $K =$ 

$K =$ 

$2K =$ 

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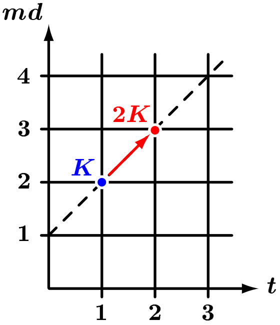

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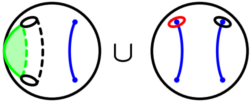
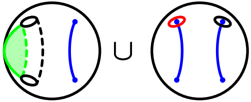
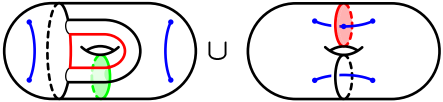
- $K =$ 

$K =$ 

$2K =$ 

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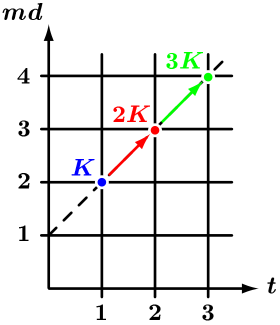

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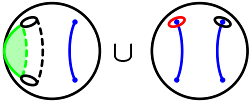
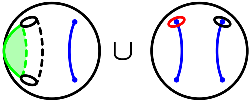
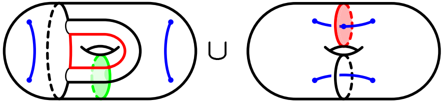
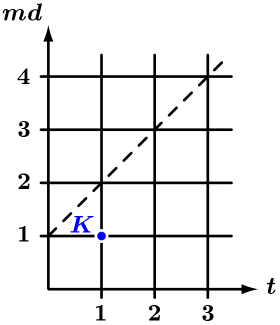
- $K =$ 

  $K =$ 

$2K =$ 

(1,1)-knots



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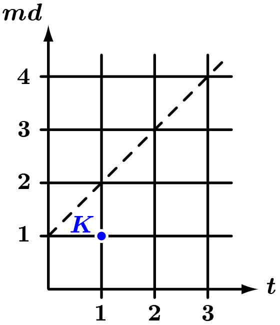
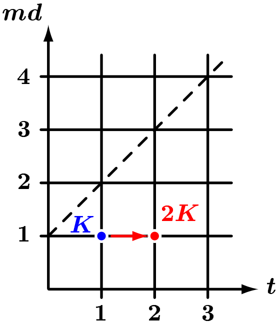


Theorem.

If K is a (1,1)-knot which is not a 2-bridge knot, then $md(2K) = 1$

(1,1)-knots



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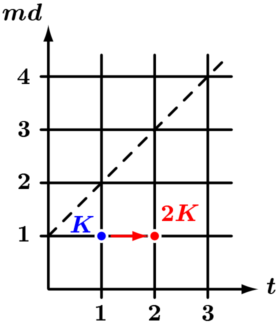

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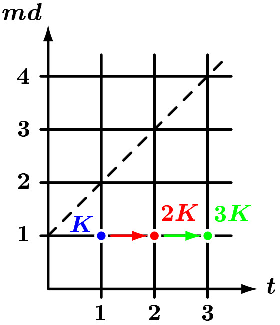

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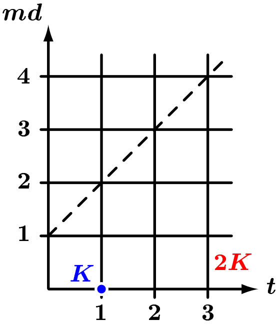
The other tunnel number one knots



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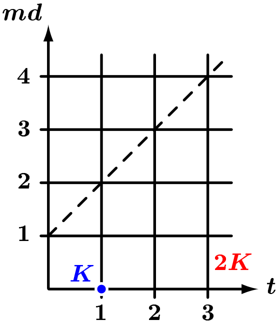

Theorem.

If K is a tunnel number one knot which is not a (1,1)-knot, then $md(2K) = 0$ or 1.

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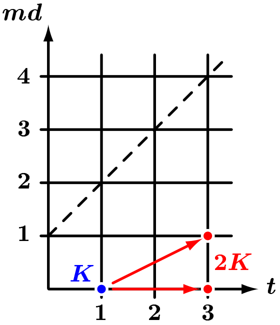

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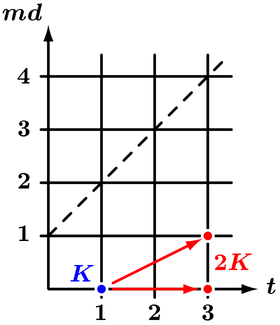

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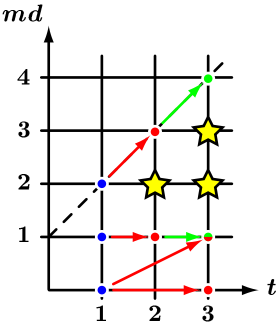
Remark.

Kobayashi-Rieck showed: $\exists K$ with $t(2K) = 3$ and $md(2K) = 0$.

Strategy of proofs



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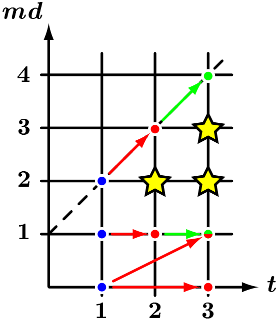
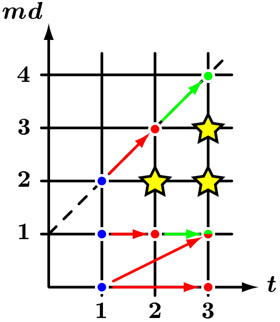


Study up on (1,2)-, (1,3)- and (2,2)-splittings.



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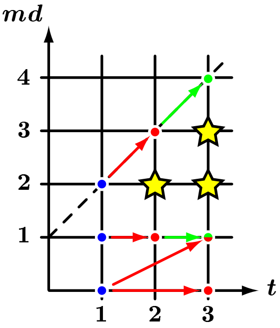
Weakly reducible case

Use *weak reduction* (by Casson-Gordon).

Strategy of proofs



Study up on (1,2)-, (1,3)- and (2,2)-splittings.

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Strongly irreducible case

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in a good position (by Schultens).
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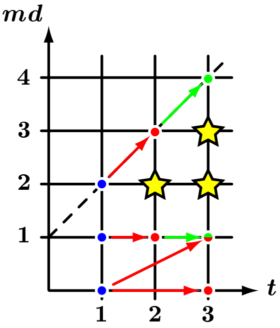

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In every situation, we can find a 2-bridge knot as a connected summand.

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Such a study would be much interesting and challenging if one takes the connected sum of knots with tunnel number greater than one, because there is a possibility of sub-additivity of tunnel number under the connected sum.