Critical Heegaard surfaces obtained by amalgamation

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GOAL

To construct critical (= topological index two) Heegaard surfaces which have new features compared to existing examples

stabilized, or unstabilized?
minimal genus, or non-minimal genus?
and so on.

• disk complex

S : a closed orientable separating surface in a 3-manifold M, dividing M into two submanifolds V and W

Define the *disk complex* $\mathcal{D}_S$ as follows.

• Vertices of $\mathcal{D}_S$ are isotopy classes of compressing disks for S.
• A collection of $k+1$ distinct vertices constitute a k-cell if there are pairwise disjoint representatives.

$\mathcal{D}_S(V)$: subcomplex of $\mathcal{D}_S$ spanned by compressing disks in V
$\mathcal{D}_S(W)$: subcomplex of $\mathcal{D}_S$ spanned by compressing disks in W

S is an *incompressible* surface
$\iff \mathcal{D}_S = \emptyset$
$\iff$ topological index 0

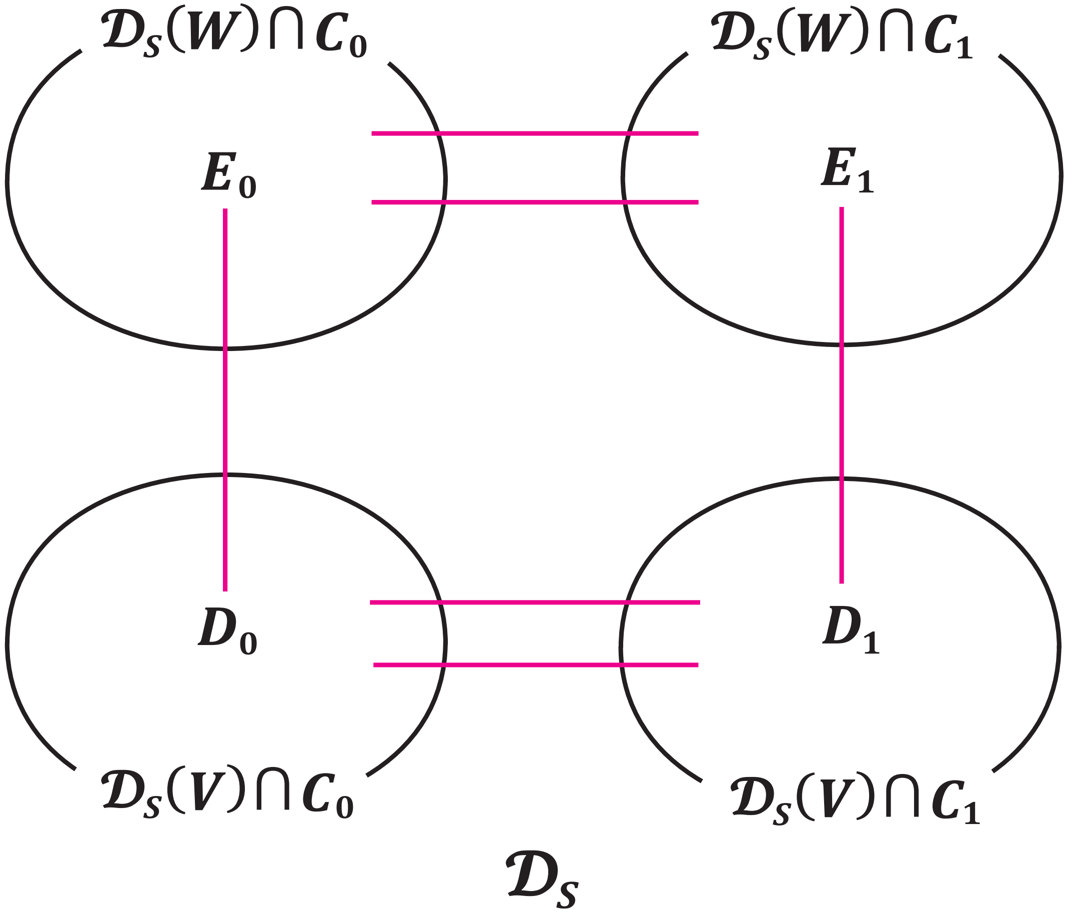
S is a *strongly irreducible* surface
$\iff$ both $\mathcal{D}_S(V)$ and $\mathcal{D}_S(W)$ are non-empty, and $\mathcal{D}_S(V)$ is not connected to $\mathcal{D}_S(W)$ in $\mathcal{D}_S$.
$\iff$ topological index 1 ($\pi_0(\mathcal{D}_S)$ is non-trivial.)

• critical surface

S is a *critical* surface if vertices of $\mathcal{D}_S$ can be partitioned into two non-empty sets C_0 and C_1.

• For each $i = 0, 1$, $\exists$ **at least one** pair of compressing disks $D_i \in \mathcal{D}_S(V) \cap C_i$ and $E_i \in \mathcal{D}_S(W) \cap C_i$ such that $D_i \cap E_i = \emptyset$.

• If $D \in \mathcal{D}_S(V) \cap C_i$ and $E \in \mathcal{D}_S(W) \cap C_{1-i}$, then $D \cap E \neq \emptyset$ for any representative disks, i.e. D and E are **not** joined by an edge.



critical
$\Longleftrightarrow$ topological index 2 ($\pi_1(\mathcal{D}_S)$ is non-trivial.)

Topologically minimal (incompressible, strongly irreducible, critical, ...) surface has nice properties.

For example,
if an irreducible 3-manifold contains an incompressible surface F
and a topologically minimal surface S,
then F and S can be isotoped so that any loop of $F \cap S$ is essential on both surfaces.

- **Known results**

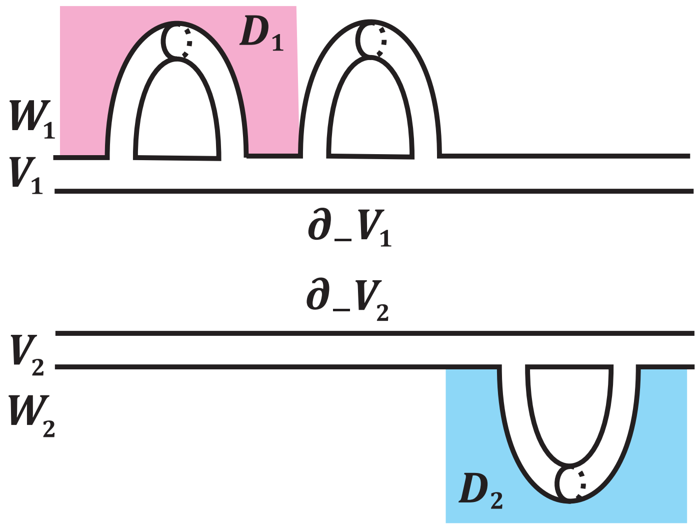
[Bachman-Johnson]
There exist arbitrarily high index topologically minimal surfaces.

[Lee]
There are unstabilized critical Heegaard surfaces.
(e.g. closed surface $\times S^1$)

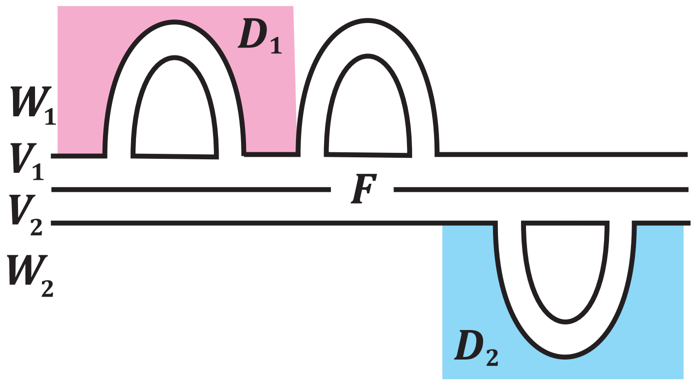
[Reidemeister], [Singer]
Any two Heegaard splittings of a 3-manifold become isotopic after a finite number of stabilizations.

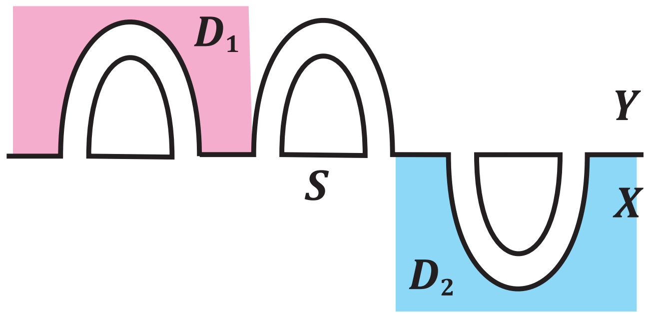
Theorem [Bachman]
If a manifold which does **not** contain incompressible surfaces has two **distinct** strongly irreducible Heegaard splittings,
then the minimal genus common stabilization of the two splittings is **critical**.

- **amalgamation**

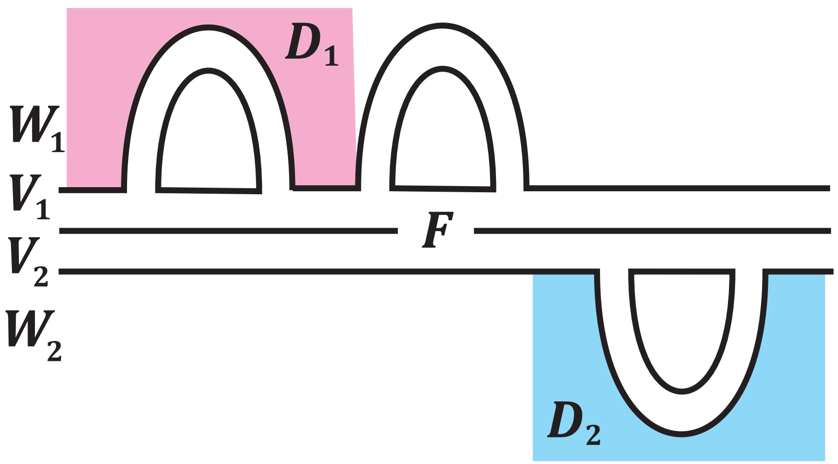


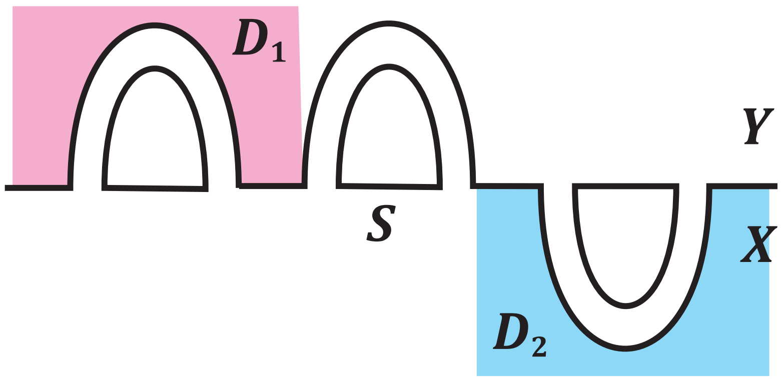
$\partial_- V_1 \simeq \partial_- V_2 \simeq F$



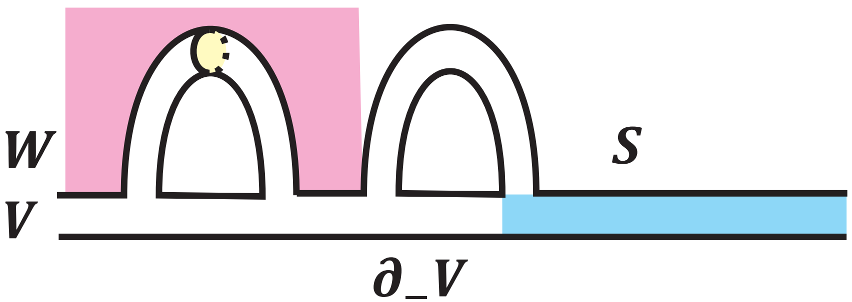


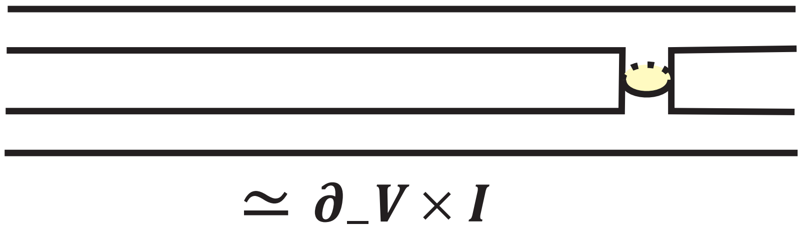
Theorem 1
Let $X \cup_S Y$ be an amalgamation of two strongly irreducible Heegaard splittings $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ along homeomorphic boundary components of $\partial_- V_1$ and $\partial_- V_2$.
Assume that V_2 is constructed from $\partial_- V_2 \times I$ by attaching **only one** 1-handle.
If $\exists$ essential disks $D_1 \subset W_1$ and $D_2 \subset W_2$ which **persist** into **disjoint** essential disks in Y and X respectively, then S is **critical**.

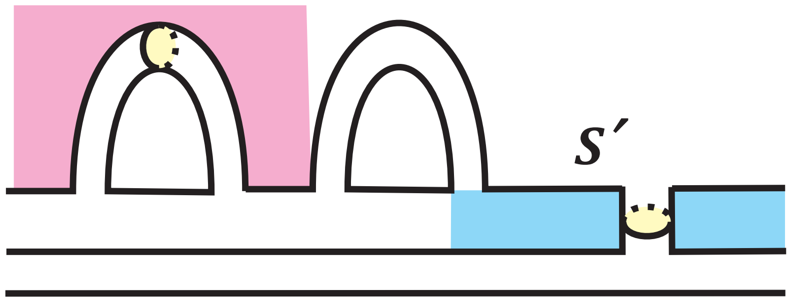




- **boundary stabilization**



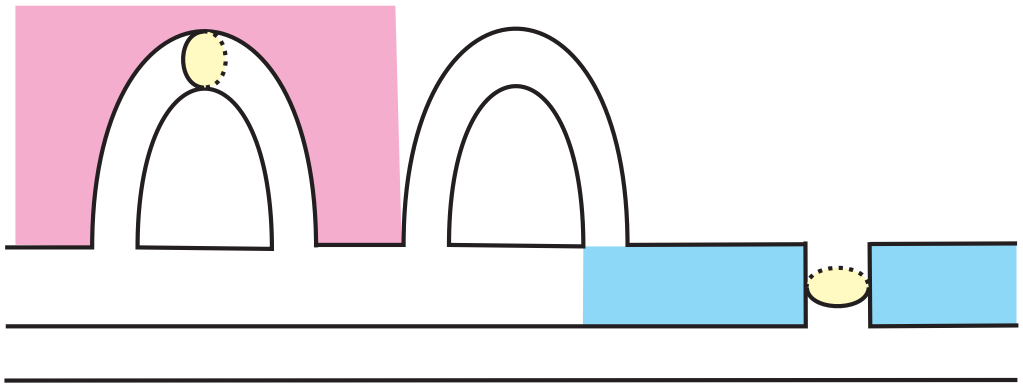




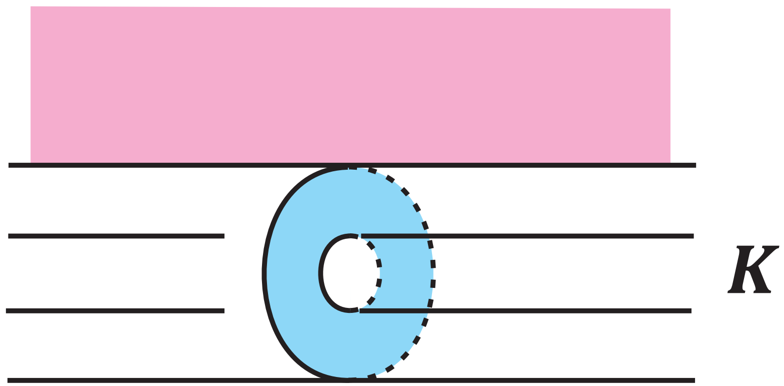
$g(S') = g(S) + g(\partial_{-}V)$
The manifold is unchanged.

Theorem 2

If a strongly irreducible Heegaard surface of a 3-manifold with boundary admits a **disjoint** pair of a vertical annulus and an essential disk on opposite sides, then the Heegaard splitting obtained from it by a boundary stabilization is **critical**.



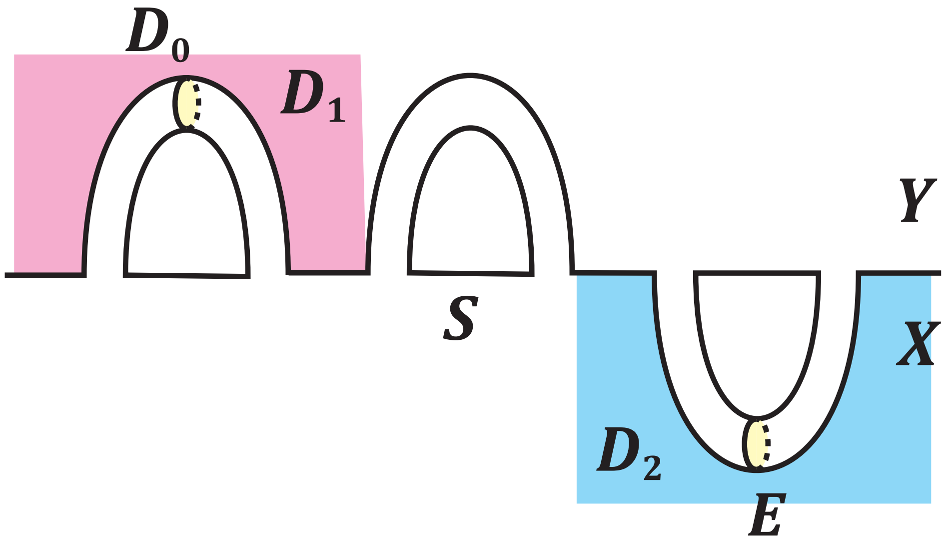
For example,
for a Heegaard splitting of the exterior of a knot K in S^3,
if a meridian of K is primitive (**primitive meridian**),
then we can take a disjoint pair of a vertical annulus and an essential disk on opposite sides.



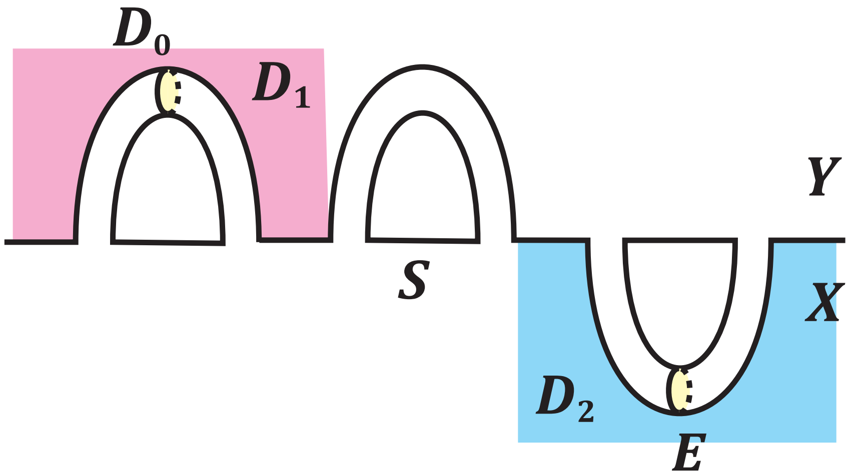
• **Remark**

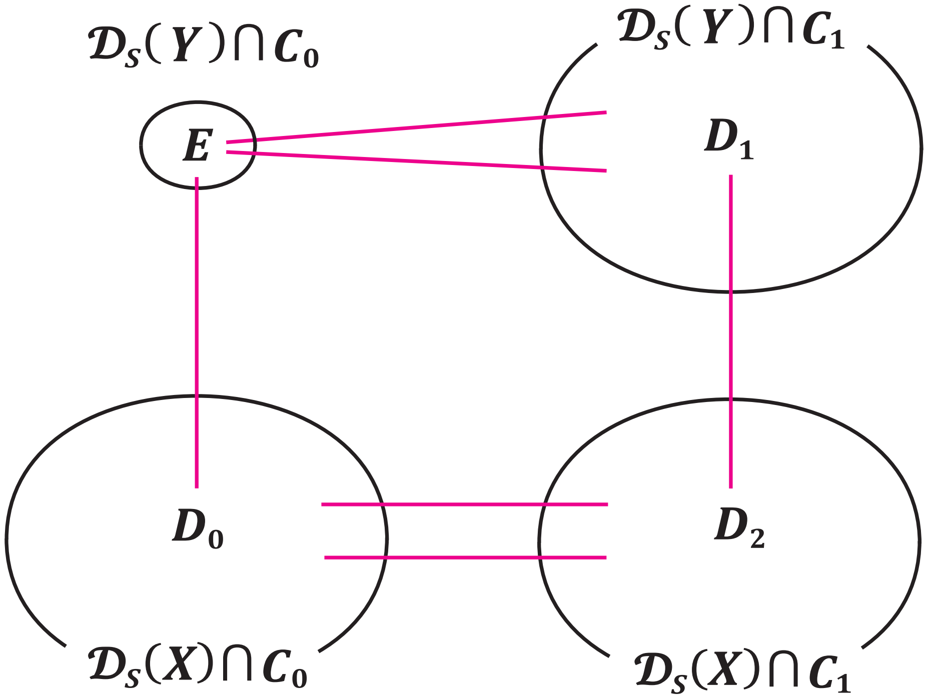
Theorem 2 gives examples of critical Heegaard surfaces of **non-minimal genus**, for 3-manifolds which do **not** admit **distinct** Heegaard splittings,
i.e. the existence of a critical Heegaard surface of **non-minimal genus** for a 3-manifold admitting a **unique** minimal genus Heegaard splitting.

- **idea of pf of Thm 1 (partition of disk complex)**



1. Let $\mathcal{D}_S(X) \cap C_0$ be essential disks in X disjoint from E.
2. Let $\mathcal{D}_S(Y) \cap C_0$ be a singleton $\{E\}$.
3. Let $\mathcal{D}_S(X) \cap C_1$ be $\mathcal{D}_S(X) - (\mathcal{D}_S(X) \cap C_0)$.
4. Let $\mathcal{D}_S(Y) \cap C_1$ be $\mathcal{D}_S(Y) - \{E\}$.


D_0
D_1
Y
S
X
D_2
E


$\mathcal{D}_S(Y) \cap C_0$
$\mathcal{D}_S(Y) \cap C_1$
E
D_1
D_0
D_2
$\mathcal{D}_S(X) \cap C_0$
$\mathcal{D}_S(X) \cap C_1$