A presentation of the fundamental biquandle of the n-twist spun trefoil

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{ quandles } ⊂ { biquandles }

A classical link → The fundamental quandle
(a complete invariant)

A surface link → The fundamental quandle
(not a complete invariant)
→ The fundamental biquandle
↑
a stronger invariant ?

$$\{ \text{ quandles } \} \subset \{ \text{ biquandles } \}$$

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Definition (biquandle)

A ***biquandle*** *is a set B with four binary operations $(a, b) \mapsto a^b, a_b, a^{\bar{b}}$ and $a_{\bar{b}}$ which satisfy the following axioms:*

1. $\forall b \in B$, *each maps* $a \mapsto a^b, a_b, a^{\bar{b}}$ *and* $a_{\bar{b}}$ *are bijections taking* $a \in B$ *into* B.
2. $\forall a,\ x,\ y \in B$,
   $x = a_x \Leftrightarrow a = x^a, \quad y = a^{\bar{y}} \Leftrightarrow a = y_{\bar{a}}$.
3. $\forall a,\ b \in B$,
   $a = a^{b\overline{b_a}}, b = b_{a\overline{a^b}}, a = a^{\bar{b}b_{\bar{a}}}, b = b_{\bar{a}a^{\bar{b}}}$.
4. $\forall a,\ b,\ c \in B$,
   $a^{bc} = a^{c_b b^c}, c_{ba} = c_{a^b b_a}, (b_a)^{c_{a^b}} = (b^c)_{a^{c_b}}$.

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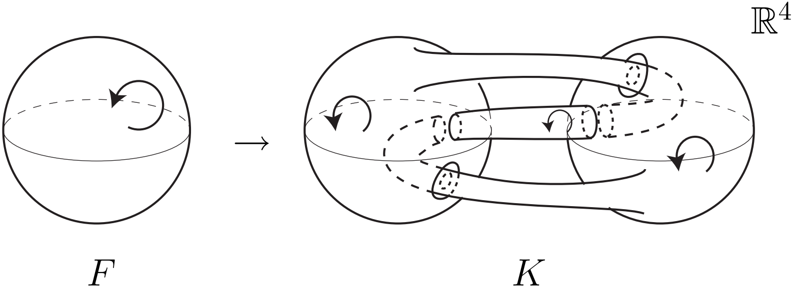
Definition (surface link)

A *surface link* K *is the image of an embedding of a closed oriented surface* F *into* $\mathbb{R}^4$.



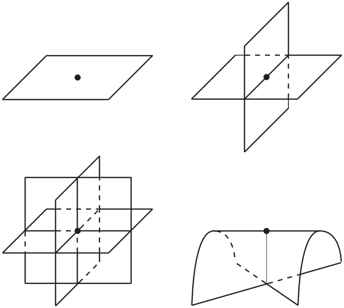
Introduction

Definition (surface diagram)

K : *a surface link.*

$p : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ *a generic projection.*

A surface diagram of K is $p(K)$ with height information.



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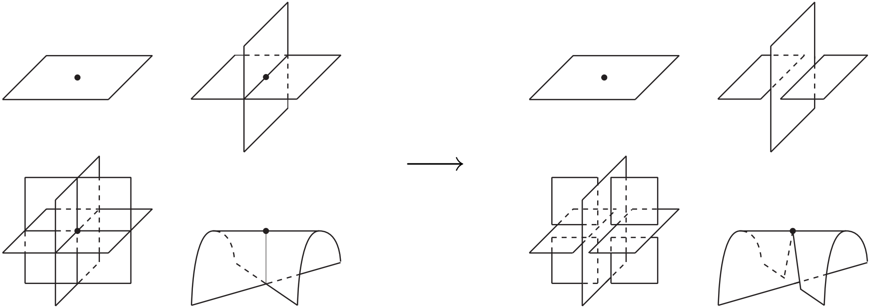
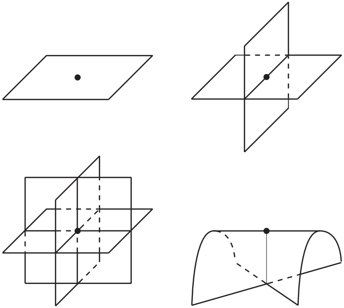


Definition (semi-sheet of a surface diagram)

D : *a surface diagram.*

Σ : *the double curves of* D.

We call connected components of $D \setminus N(\Sigma)$ *semi-sheets of* D.



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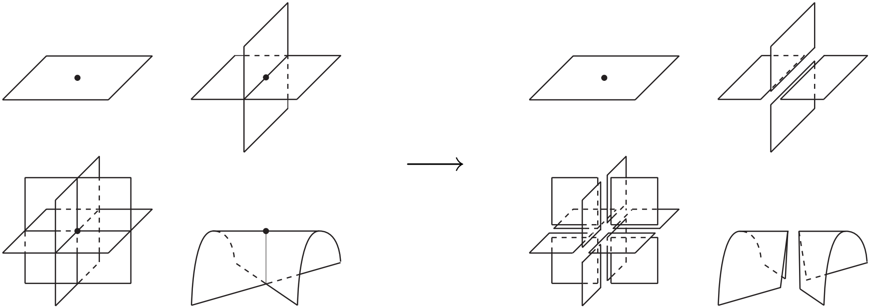

K : a surface link. $\rightarrow$ $BQ(K)$: the fundamental biquandle of K.

$\downarrow$ $\quad\quad\quad\quad$ $\Updownarrow$ def

D : a surface diagram. $\rightarrow$ $BQ(D)$: the fundamental biquandle of D.

Theorem (Carrell'09)

The definition of $BQ(K)$ is well-defined.

The biquandle $BQ(K)$ is an invariant of K.

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Theorem (Carrell'09)

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ch-diagram

Definition (ch-diagram)

Γ : *a singular link diagram with some markers.*
We call Γ *a ch-diagram.*

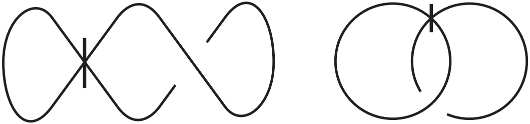
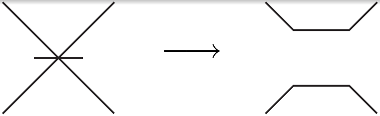


Figure: examples of ch-diagrams

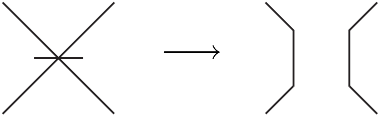
A ch-diagram with a certain condition represents a surface link.

Definition (adequate ch-diagram)

A ch-diagram is adequate if a diagram of the unlink results when all marked vertices are consistently A- or B-smoothed.



A-smoothing

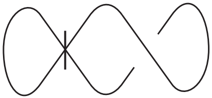


B-smoothing

Definition (surface link constructed by a ch-diagrm)

Γ : *an adequate ch-diagram.*

$F(\Gamma) \subset \mathbb{R}^4$: *the surface link constructed as follows:*



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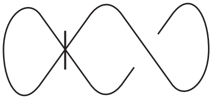

$\longrightarrow \quad t = 0$

Γ $\qquad$ $F(\Gamma)$

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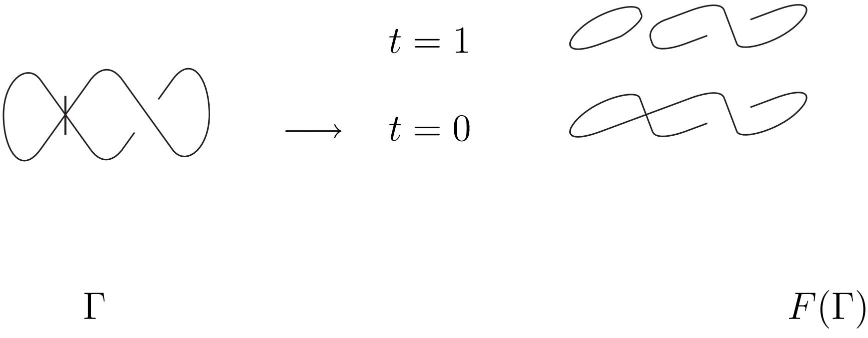

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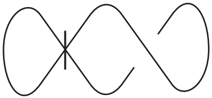
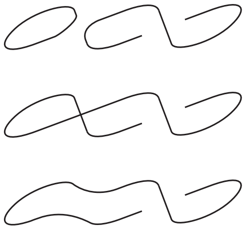


$\longrightarrow$

$t = 1$

$t = 0$

$t = -1$



Γ $F(\Gamma)$

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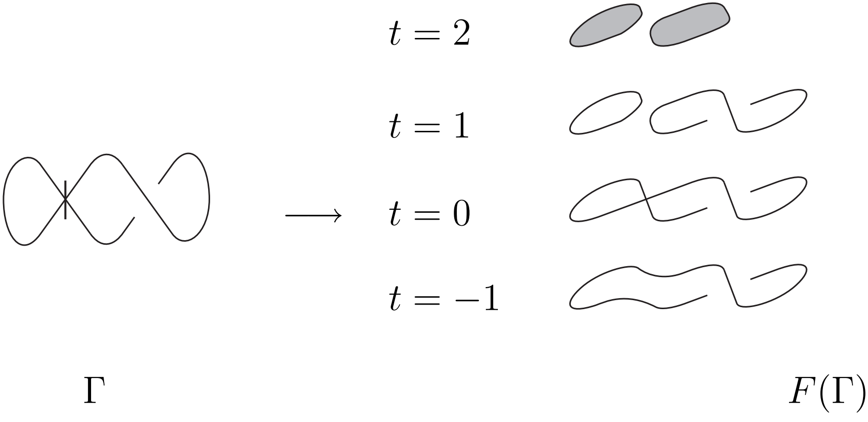

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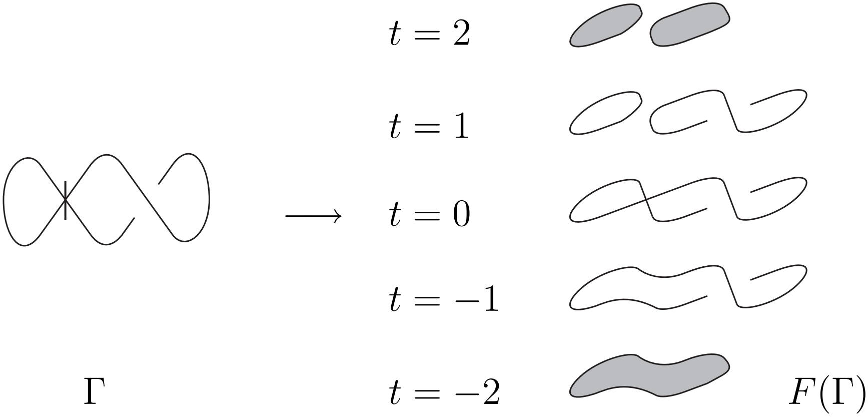

ch-diagram

Definition (ch-diagram of a surface link)

K : a surface link.
An adequate ch -diagram Γ is a ch-diagram of K if $F(\Gamma) \cong K$ in $\mathbb{R}^4$.

Theorem (Yoshikawa'94)

Any surface link K has a ch-diagram of K.

Definition and Theorem

Γ : an oriented adequate. $\rightarrow$ $BQ(\Gamma)$: the fundamental biquandle of Γ.

$\downarrow$

$F(\Gamma)$: the surface. associated to Γ

$\updownarrow \cong$

K : a surface link. $\rightarrow$ $BQ(K)$

Theorem (A.)

K : a surface link.
Γ : a ch-diagram of K.
Then $BQ(K) \cong BQ(\Gamma)$.

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Theorem (A.)

K : a surface link.

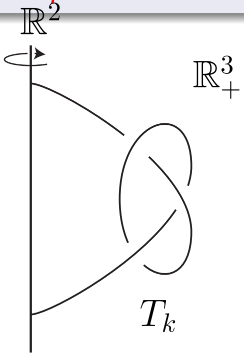
Γ : a ch-diagram of K.

Then $BQ(K) \cong BQ(\Gamma)$.

The spun trefoil

Definition (spun knot)

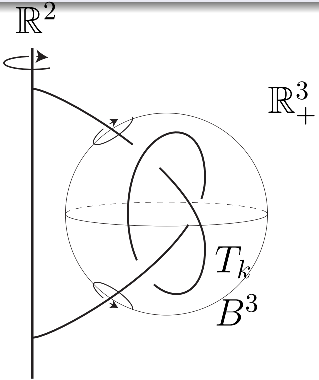
$\mathbb{R}^3_+ := \{(x_1, x_2, x_3, x_4) \mid x_3 \geq 0, x_4 = 0\}$.
k *: a classical knot.*
$T_k \subset \mathbb{R}^3_+$ *: a tangle of* k *whose end points lie in* $\mathbb{R}^2$*.*
Then $\{(x_1, x_2, x_3 \sin\theta, x_3 \cos\theta) \mid (x_1, x_2, x_3) \in T_k, \theta \in [0, 2\pi)\}$ *forms a surface knot. We call it the spun* k*.*



The twist-spun trefoil

Definition (twist-spun knot)

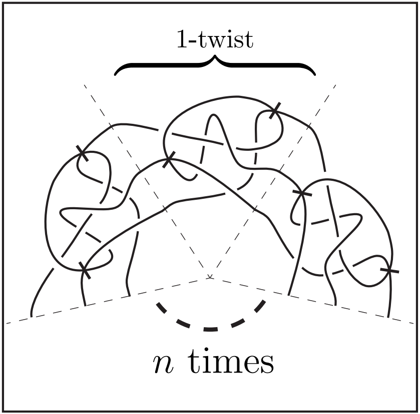
While rotating $\mathbb{R}^3_+$ about $\mathbb{R}^2$, twist B^3 n-times ($n \in \mathbb{Z}$). Then we have another surface knot. We call it the n-twist-spun k.



The twist-spun trefoil

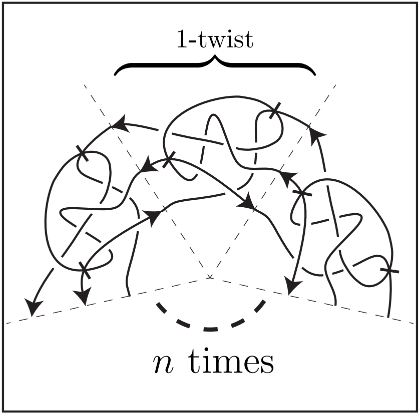
Theorem (Inoue)

The following ch-diagram represents the n-twist-spun trefoil.



Main Theorem

$\tau^n T$: the oriented ch-diagram as illustrated in below, then we have the following result.



Main Theorem

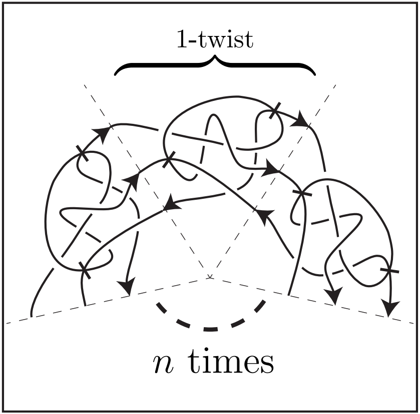
$$\phi_1(a,b,c) = (b^{\bar{b}^{-1}})_a$$
$$\phi_2(a,b,c) = b^{\bar{b}^{-1}\overline{c^b}}$$
$$\phi_3(a,b,c) = b_c$$
$$r(a,b,c) : a^{b^{\bar{b}^{-1}}} = (c^b)_{\overline{b^{\bar{b}^{-1}}}}$$

Theorem (A.)

$BQ(\tau^n T)$
$= \langle a_1, b_1, c_1, \ldots, a_n, b_n, c_n \mid a_{i+1} = \phi_1(a_i, b_i, c_i), b_{i+1} = \phi_2(a_i, b_i, c_i), c_{i+1} = \phi_3(a_i, b_i, c_i), r(a_i, b_i, c_i), a_1 = \phi_1(a_n, b_n, c_n), b_1 = \phi_2(a_n, b_n, c_n), c_1 = \phi_3(a_n, b_n, c_n)\rangle.$

Main Theorem

$-\tau^n T$: the orientation reversed $\tau^n T$,
then we have the following result.



Main Theorem

$$\psi_1(a,b,c) = (b^{b^{-1}})_{\bar{a}}$$
$$\psi_2(a,b,c) = b^{b^{-1}c^{\bar{b}}}$$
$$\psi_3(a,b,c) = b_{\bar{c}}$$
$$r'(a,b,c) : a^{\overline{b^{b^{-1}}}} = (c^{\bar{b}})_{b^{b^{-1}}}$$

Theorem (A.)

$BQ(-\tau^n T)$
$= \langle a_1, b_1, c_1, \ldots, a_n, b_n, c_n \mid a_{i+1} = \psi_1(a_i, b_i, c_i), b_{i+1} = \psi_2(a_i, b_i, c_i), c_{i+1} = \psi_3(a_i, b_i, c_i), r'(a_i, b_i, c_i), a_1 = \psi_1(a_n, b_n, c_n), b_1 = \psi_2(a_n, b_n, c_n), c_1 = \psi_3(a_n, b_n, c_n) \rangle.$

Thank you!