On the n-shake genus of a knot

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Today's contents

1. The 4-ball genus and n-shake genus
2. On the n-shake genus ($n \neq 0$)
3. On the 0-shake genus

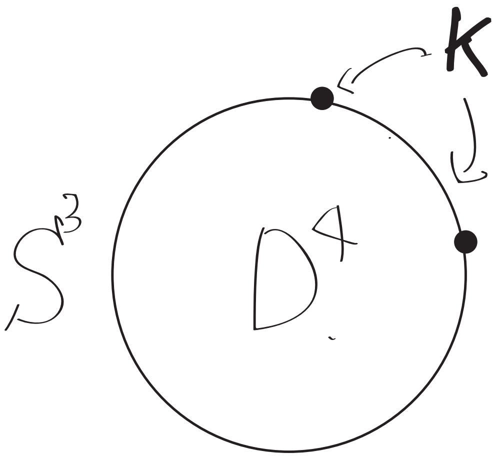
The 4-ball genus of a knot

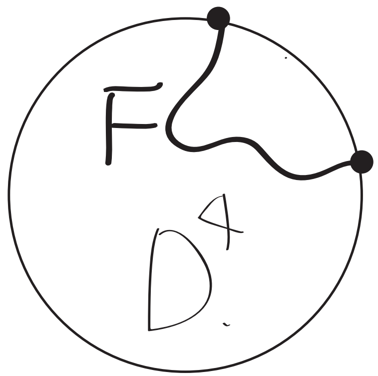
K : a knot in $S^3 = \partial D^4$

The 4-ball genus and a slice knot

$$g_*(K) := \min\{g(F) | F \subset D^4, \partial F = K.\}$$

A knot K is slice iff $g_*(K) = 0$.





The n-shake genus of a knot

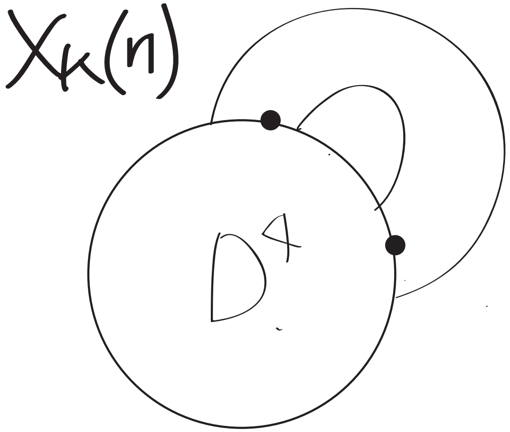
K : a knot in $S^3 = \partial D^4$

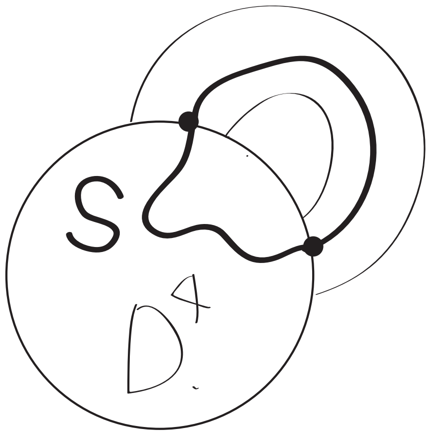
$X_K(n)$: $D^4 \cup$ (a 2-handle)

(The attaching sphere is K with framing n)

The n-shake genus

$g_n^s(K) := \min\{g(S) | [S] \text{ generates } H_2(X_K(n)) \simeq \mathbb{Z}.\}$



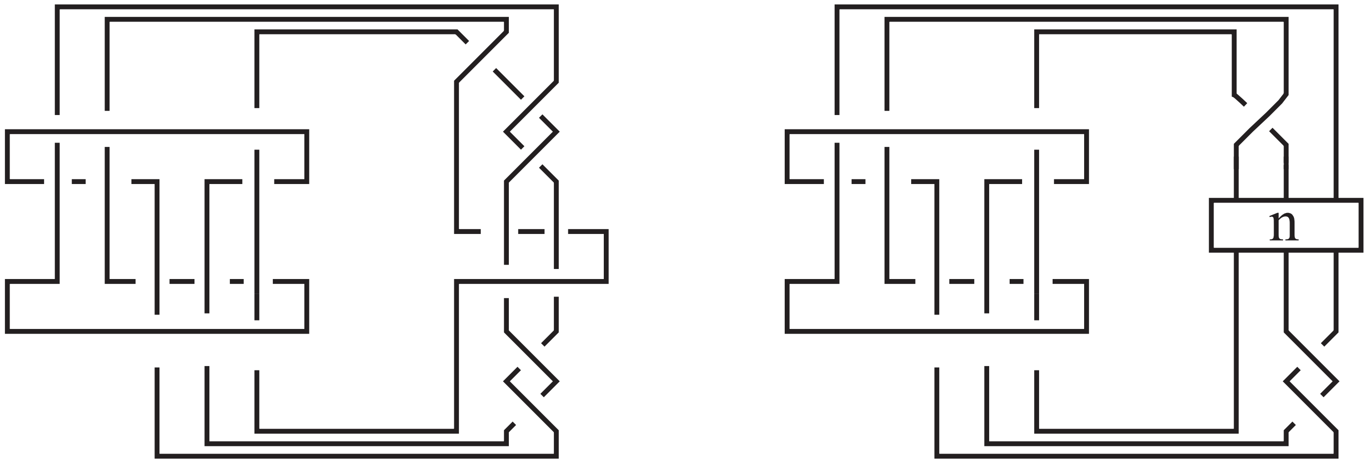


Remark

$$g_n^s(K) \leq g_*(K).$$

If K is slice, then $g_n^s(K) = 0$.

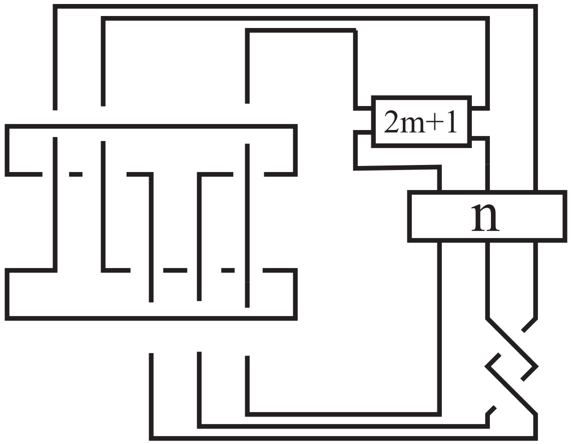
2. On the n-shake genus $(n \neq 0)$

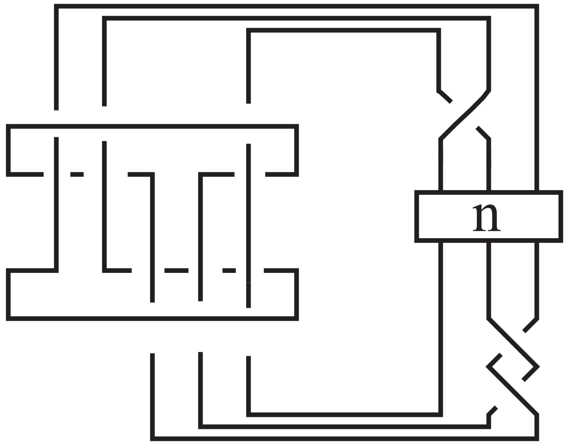
K_1  K_n

In 1977, Akbulut showed that $g_1^s(K_1) < g_*(K_1)$.

In 2010, Omae showed $g_n^s(K_n) < g_*(K_n)$ $(n \neq 0)$.

Main result

$K_n(m)$ 



$K_n(0)$

Theorem[A.].

If $n \neq 0$ and $0 \leq m$,

$$g_n^s(K_n(m)) = 0, \quad g_*(K_n(m)) = 1.$$

Corollary[A.].

For each integer $n \neq 0$,

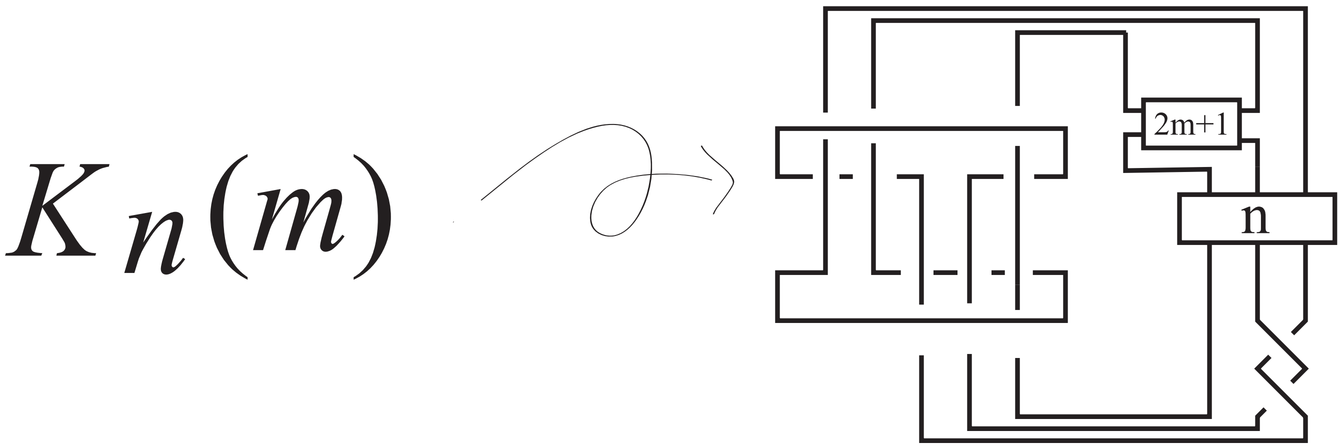
there exists ∞-many knots K with $g_n^s(K) < g_*(K)$.

Proof of Theorem

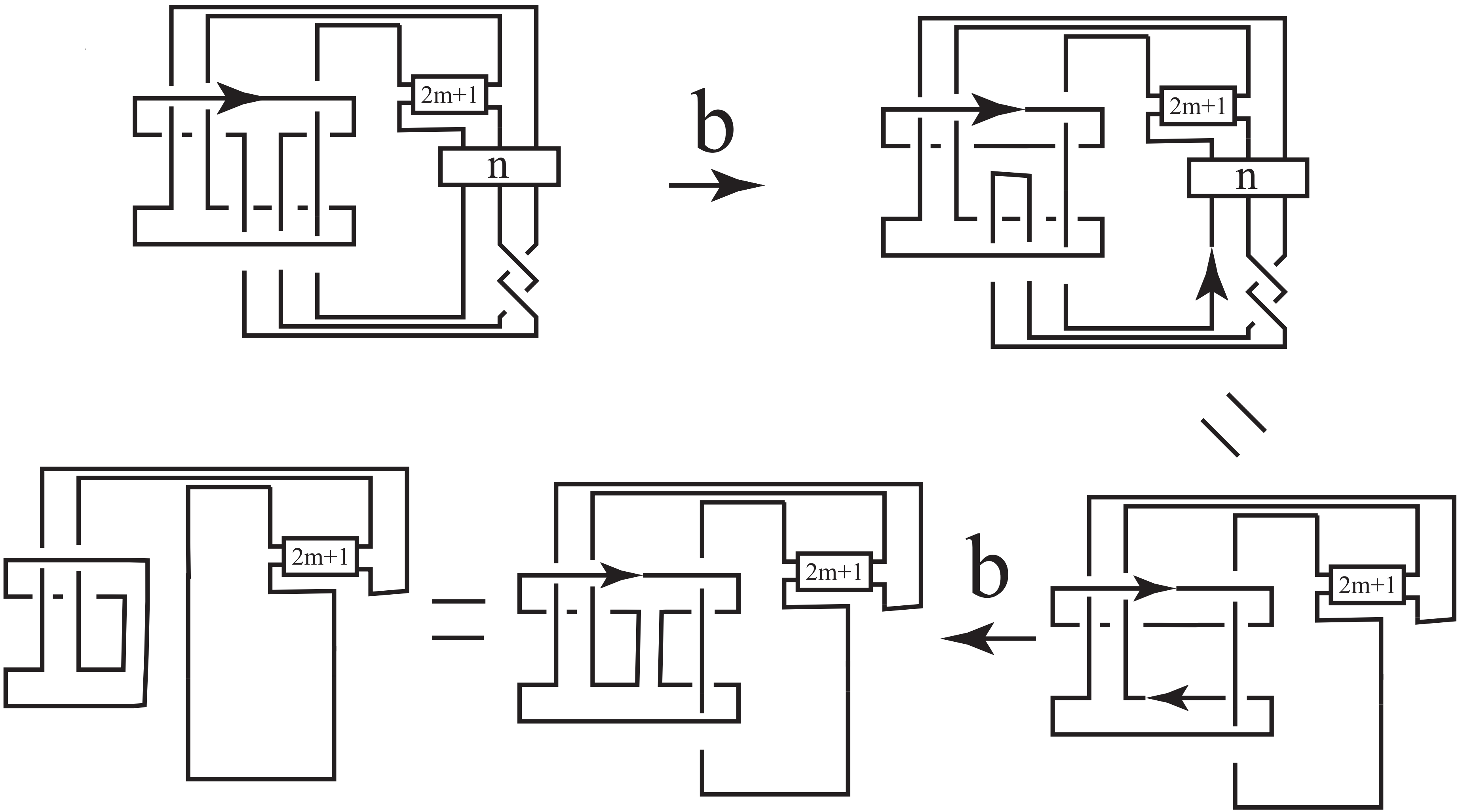
- $g_*(K_n(m)) \leq 1$ (Band surgery)

- $g_*(K_n(m)) \neq 0$ (Calculation of the Alexander poly.)

- $g_n^s(K_n(m)) = 0$ (Kirby calculus)



Proof of Theorem (Step 1)



- This implies that $g_*(K_n(m)) \leq 1$.

Proof of Theorem (Step 2)

$\Delta_K(t)$: The Conway-Alexaneder polynomial of K

Theorem[Fox-Milnor, Terasaka].

If K is slice, then $\Delta_K(t) = \exists F(t)F(t^{-1})$. Moreover, the 0-th coefficient of $\Delta_K(t)$ is positive.

Proof.

$$\begin{aligned}\Delta_K(t) =& a_0 + a_1(t+\frac{1}{t}) + a_2(t^2+\frac{1}{t^2}) + \cdots . \\ =& F(t)F(t^{-1}) \\ =& \sum_{i\geq 0} b_i t^i \sum_{i\geq 0} b_i t^{-i} \\ =& \sum_{i\geq 0} b_i^2 + a_1(t+\frac{1}{t}) + a_2(t^2+\frac{1}{t^2}) + \cdots .\end{aligned}$$

Lemma[A.].

Suppose that $n > 0$. Then

$$\Delta_{K_n(m)}(t) = -(1+6\mathrm{m}) + (2+4m)(t+\frac{1}{t}) - (1+m)(t^2+\frac{1}{t^2}) - (1+3m)(t^{n-1}+\frac{1}{t^{n-1}}) + (2+3m)(t^n+\frac{1}{t^n}) - (1+m)(t^{n+1}+\frac{1}{t^{n+1}}).$$

$$\Delta_{K_{-n}(m)}(t) = -(1+6\mathrm{m}) + (2+4m)(t+\frac{1}{t}) - (1+m)(t^2+\frac{1}{t^2}) - (1+m)(t^{n-1}+\frac{1}{t^{n-1}}) + (2+3m)(t^n+\frac{1}{t^n}) - (1+3m)(t^{n+1}+\frac{1}{t^{n+1}}) + m(t^{n+2}+\frac{1}{t^{n+2}}).$$

- $K_n(m)$ is not slice, therefore, $g_*(K_n(m)) = 1$.

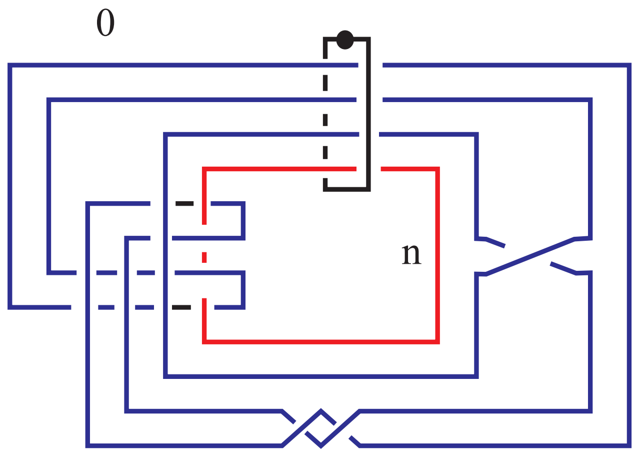
Kirby calculus

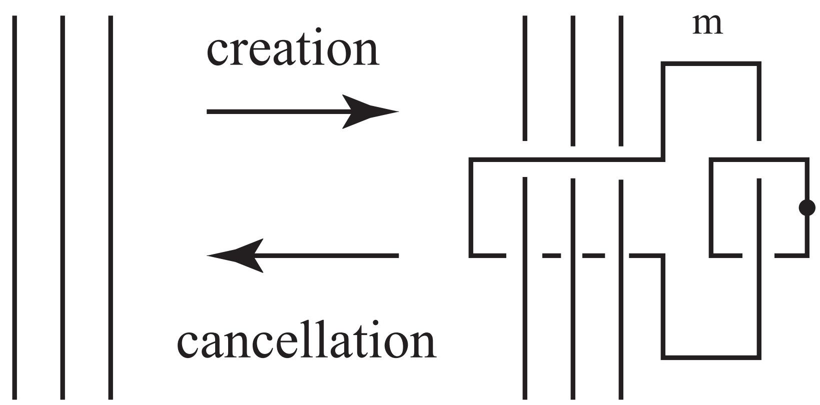
Any 4-mfd is represented by a Kirby diagram,

which are links in $\mathbb{R}^3$ with integers or dots.

The following moves preserve the diffeo type:

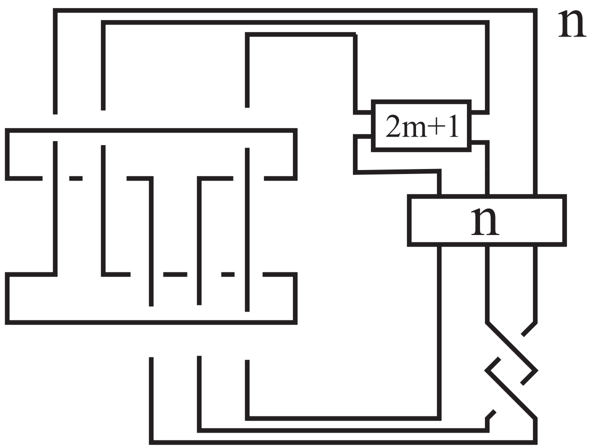
- isotopies of the links
- handle slides (which preserve the number of links)
- handle creation / cancellation

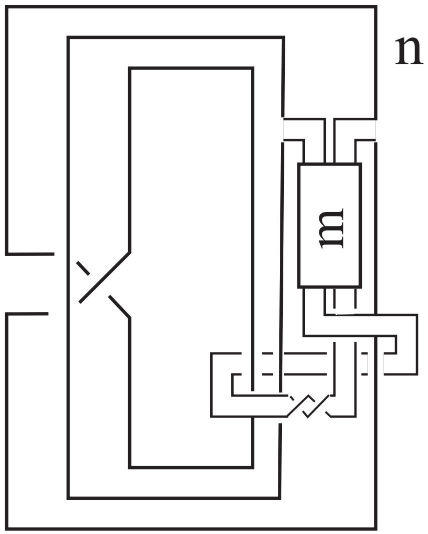




Proof of Theorem (Step 3)

$K_n(m)$

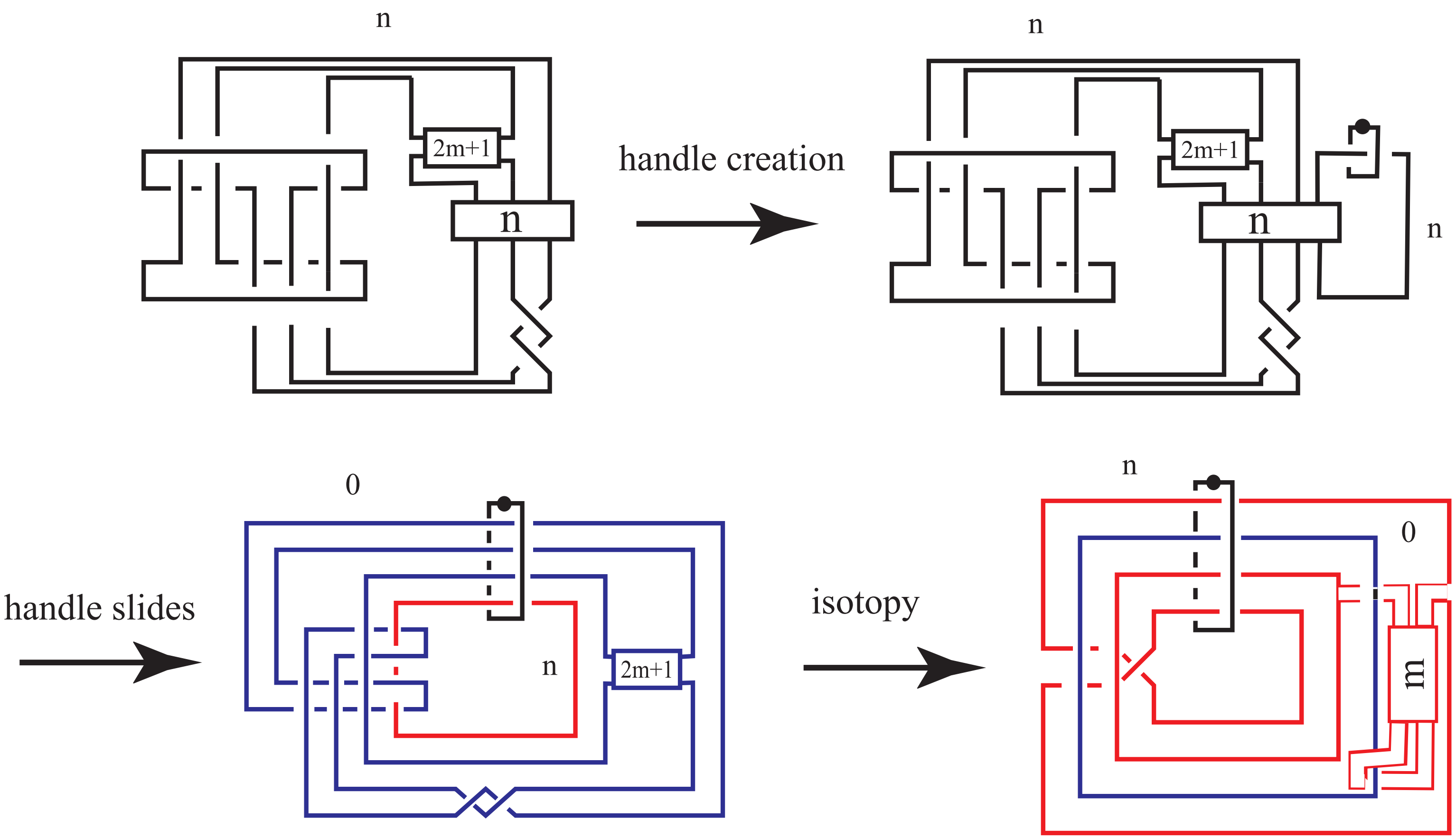


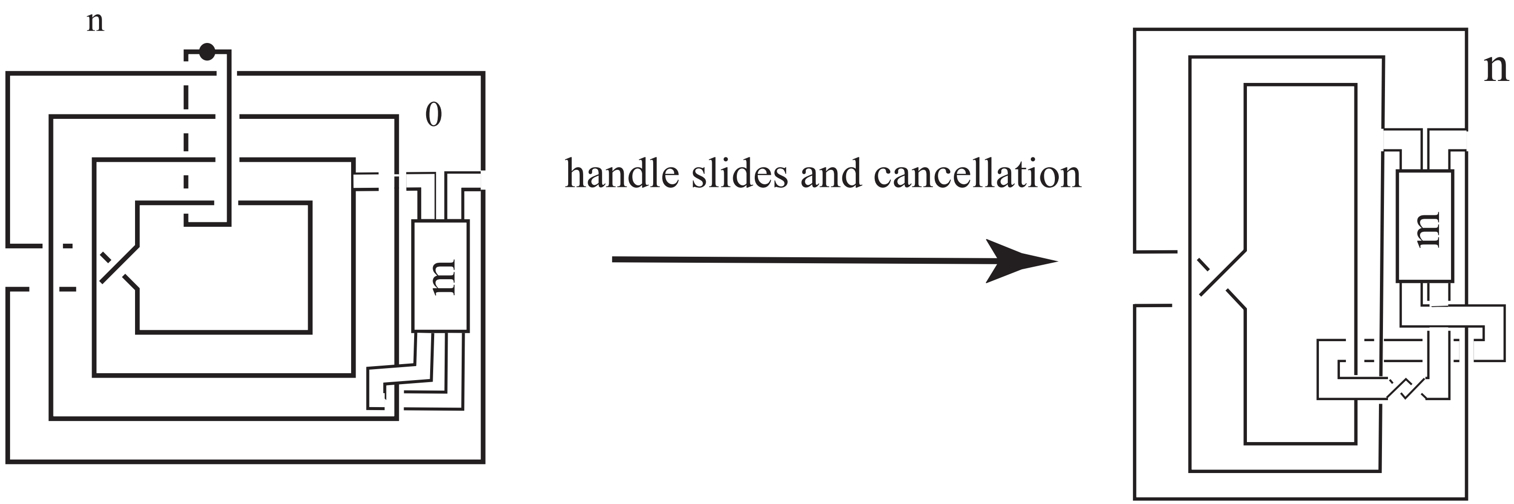


$R(m)$

Lemma[A.].

$X_{K_n(m)}(n)$ and $X_{R(m)}(n)$ are diffeomorphic.


n
2m+1
n
handle creation
n
2m+1
n
n
handle slides
0
n
2m+1
isotopy
n
0
m



Corollary[A.].

$$g_n^s(K_n(m)) = 0$$

Proof. Since $R(m)$ is ribbon, $g_n^s(R(m)) = 0$.

Since $X_{K_n(m)}(n)$ and $X_{R(m)}(n)$ are diffeomorphic,

$$g_n^s(K_n(m)) = g_n^s(R(m)).$$

3. On the 0-shake genus

Kirby's problem 1.41 (Akbulut)

$$g_0^s(K) = g_*(K).$$

Fact

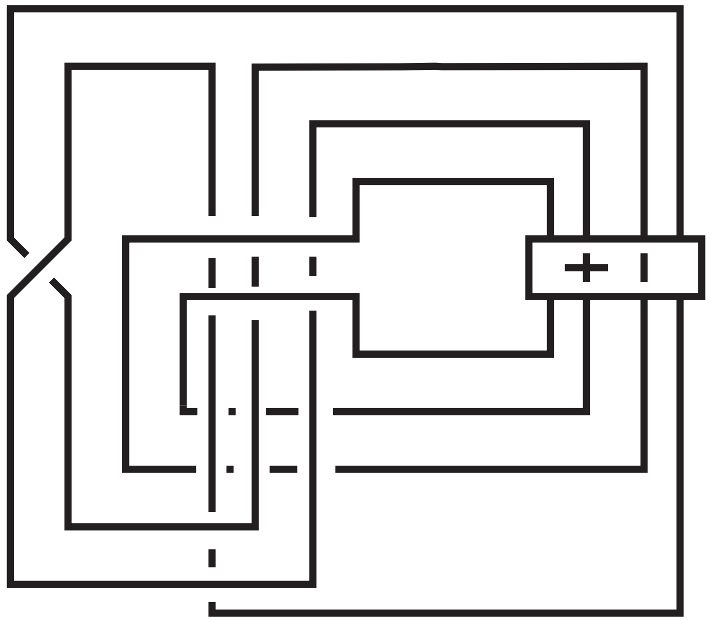
The equality holds for $(2, q)$-torus knots.

Tool. The adjunction inequality in 4-manifold theory is interpreted as follows: For a knot K and its Legendrian representative L with $\mathrm{tb}(L) \geq 1$,

$$\mathrm{tb}(L) + |\mathrm{rot}(L)| \leq g_0^s(K).$$

Omae's knots

Omae (ex-student of Endo) asked me whether the following knot is slice or not.



- The 0-shake genus of Omae's knot is zero.