On two-point homogeneous quandles

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1. Riemannian symmetric space and two-point homogeneous

(M, g): connected Riemannian manifold
d: distance function defined by g

Definition

(M, g) is a Riemannian symmetric space.
$\Longleftrightarrow$
$\forall x \in M, \exists s_x : M \to M$ s,t

- s_x is an isometry
- $s_x^2 = id_M$
- x is an isolated fixed point of s_x

X: set

Definition

Let $* : X \times X \to X$ be a binary operator. The pair $(X, *)$ is called a quandle if

- $\forall x \in X, x * x = x$,
- $\forall x, y \in X, \exists 1 z \in X$ s,t $z * x = y$,
- $\forall x, y, z \in X, (x * y) * z = (x * z) * (y * z)$.

Proposition

Every connected Riemannian symmetric space is a quandle.

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Every connected Riemannian symmetric space is a quandle.

Definition

(M, g) is said to be two-point homogeneous.

$\Longleftrightarrow$

$\forall(x_1, x_2), (y_1, y_2) \in M \times M \ (d(x_1, x_2) = d(y_1, y_2))$,

$\exists f \in \mathrm{Isom}(M, g)$ s,t $f(x_i) = y_i \ (i = 1, 2)$

{ Riem sym sp} $\supset$ { two-point homog Riem sym sp }

$\cap$

{ quandle} $\supset$ { two-point homog quandle }

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2. two-point homogeneous quandle

$(X, *)$: a quandle
$*x : X \to X \mid y \mapsto y * x$: an automorphism
$\mathrm{Inn}(X, *) := < *x \ (\ x \in X) >$: innerautomorphism group

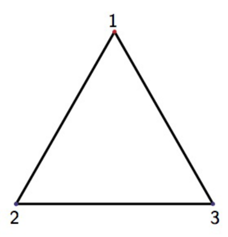
Definition

A quandle $(X, *)$ is called a two-point homogeneous quandle.
$\iff$
$\forall (x_1, x_2), (y_1, y_2) \in X \times X \ (x_1 \neq x_2, y_1 \neq y_2)$
$\exists f \in \mathrm{Inn}(X, *)$ s,t $f(x_i) = y_i \ \ (i = 1, 2)$

Example

- *The dihedral quandle with order 3 is a two-point homogeneous quandle.*

dihedral quandle with order 3



$$*1 = (2,3), *2 = (1,3), *3 = (1,2)$$

Remark: The dihedral quandle with order$\geq$ 4 is NOT a two-point homogeneous quandle.

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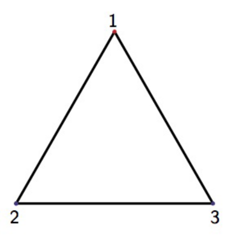

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Remark: The dihedral quandle with order$\geq$ 4 is NOT a two-point homogeneous quandle.

Definition

A quandle $(X, *)$ with order n said to be of cyclic type.

$\Longleftrightarrow$

For every $x \in X$, $*x$ acts on $X - \{x\}$ as a cyclic permutation of order $n - 1$.

Example

- *The dihedral quandle with order 3 is of cyclic type.*
- *The tetrahedron quandle is of cyclic type.*

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A quandle $(X, *)$ with order n said to be of cyclic type.

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For every $x \in X$, $*x$ acts on $X - \{x\}$ as a cyclic permutation of order $n - 1$.

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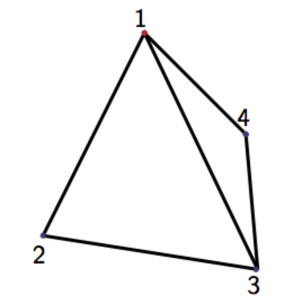
tetrahedron quandle



$$*1 = (2,3,4), *2 = (1,4,3), *3 = (1,2,4), *4 = (1,3,2)$$

Proposition

Every quandle of cyclic type is two-point homogeneous quandle.

Proposition

$(X,*)$: *quandle* ($\#X=p+1$, p: *prime*)
$(X,*)$ *is two-point homogeneous* $\Rightarrow$ $(X,*)$ *is of cyclic type*

Q, cyclic type $\Leftrightarrow$ two-point homogeneous ?

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3. Classification of quandles of cyclic type

$n \geq 3$
$s_1 := (2, 3, 4, ..., n) \in S_n$
D_n is the subset of S_n whose element s satisfies

- $s(2) = 2$
- $\{s^m s_1 s^{-m} | m = 1, 2, ..., n-2\} = \{s_1^m s s_1^{-m} | m = 1, 2, ..., n-2\}$
- s is a cyclic permutation of order $n-1$.

$A_n := \{(X, *) \text{ cyclic type} | \#X = n\}/\text{isom}$

Theorem [Tamaru]

There is a bijection $A_n \longrightarrow D_n$.

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Theorem [Tamaru]

There is a bijection $A_n \longrightarrow D_n$.

Classification of quandles of cyclic type

- $n = 3$

  $D_3 = \{(1,3)\}$
- $n = 4$

  $D_4 = \{(1,4,3)\}$
- $n = 5$

  $D_5 = \{(1,3,5,4),(1,4,3,5)\}$
- $n = 6$

  $D_6 = \emptyset$
- $n = 7$

  $D_7 = \{(1,7,4,6,5,3),(1,7,5,4,6,3)\}$

Classification of quandles of cyclic type

- $n = 8$

  $D_8 = \{(1, 5, 8, 3, 7, 6, 4), (1, 7, 5, 4, 8, 3, 6)\}$

- $n = 9$

  $D_9 = \{(1, 4, 3, 8, 6, 9, 5, 7), (1, 5, 7, 3, 6, 4, 9, 8)\}$

- $n = 10$

  $D_{10} = \emptyset$

- $n = 11$

  $$D_{11} = \{(1, 3, 6, 8, 4, 11, 5, 10, 9, 7), (1, 4, 3, 7, 10, 5, 11, 9, 6, 8), (1, 6, 8, 5, 3, 9, 4, 7, 11, 10), (1, 7, 5, 4, 9, 3, 10, 6, 8, 11)\}$$

- $n = 12$

  $D_{12} = \emptyset$

4. Problems

- Every quandle of cyclic type is two-point homogeneous quandle.
- For any odd prime number p, there exists a two-point homogeneous quandle with order p and it is of cyclic type.

Problem

Is there a quandle that is two-point homogeneous quandle but is not of cyclic type?

Problem

- Application to knot theory
- Calculation of quandle cohomology groups of two-point homogeneous quandles and quandles of cyclic type
- Decide of conditions of n when a quandle of cyclic type exists

Thank you very much