Finiteness of Integral Points in Relative Moduli Spaces of The One-Holed Torus

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The $\mathrm{SL}(2,\mathbb{C})$ Character Variety

An $\mathrm{SL}(2,\mathbb{C})$ **character** of the one-holed torus $T_{1,1}$ is an $\mathrm{SL}(2,\mathbb{C})$-conjugacy class of representations (i.e., homomorphisms)

$$\rho : \pi_1(T_{1,1}) \longrightarrow \mathrm{SL}(2,\mathbb{C}).$$

Note. $\pi_1(T_{1,1})$ is a free group of rank two.

An (unoriented) **simple closed curve** in $\mathsf{T}_{1,1}$ = the isotopy class of such a curve.

Generating pair = Two (unoriented) s.c.c.'s in $\mathsf{T}_{1,1}$ which cut in 1 point.

$a, b \in \pi_1(\mathsf{T}_{1,1})$ generate $\pi_1(\mathsf{T}_{1,1})$ $\Leftrightarrow$ they are the homotopy classes of a generating pair.

A generating pair $\leftrightsquigarrow$ Four pairs $\{a^{\pm 1}, b^{\pm 1}\}$ of simultaneous conjugacy classes in $\pi_1(\mathsf{T}_{1,1})$.

Furthermore, each of $\{a, ab\}$ and $\{ab, b\}$ is also such a pair of classes in $\pi_1(\mathsf{T}_{1,1})$.

Every pair of simultaneous conjugacy classes of free generators of $\pi_1(\mathsf{T}_{1,1})$ can be obtained by repeatedly obtaining new pairs in this way.

Generating triple = a triple of unoriented s.c.c.'s in $\mathsf{T}_{1,1}$ which are pairwise generating.

One gen. pair $\rightsquigarrow$ two gen. triples. Given a generating pair $\{a, b\}$, there are exactly two s.c.c.'s, ab and ab^{-1}, such that $\{a, b, ab\}$ and $\{a, b, ab^{-1}\}$ are generating triples.

Peripheral s.c.c. in $T_{1,1}$ $\leftrightsquigarrow$ $aba^{-1}b^{-1}$.

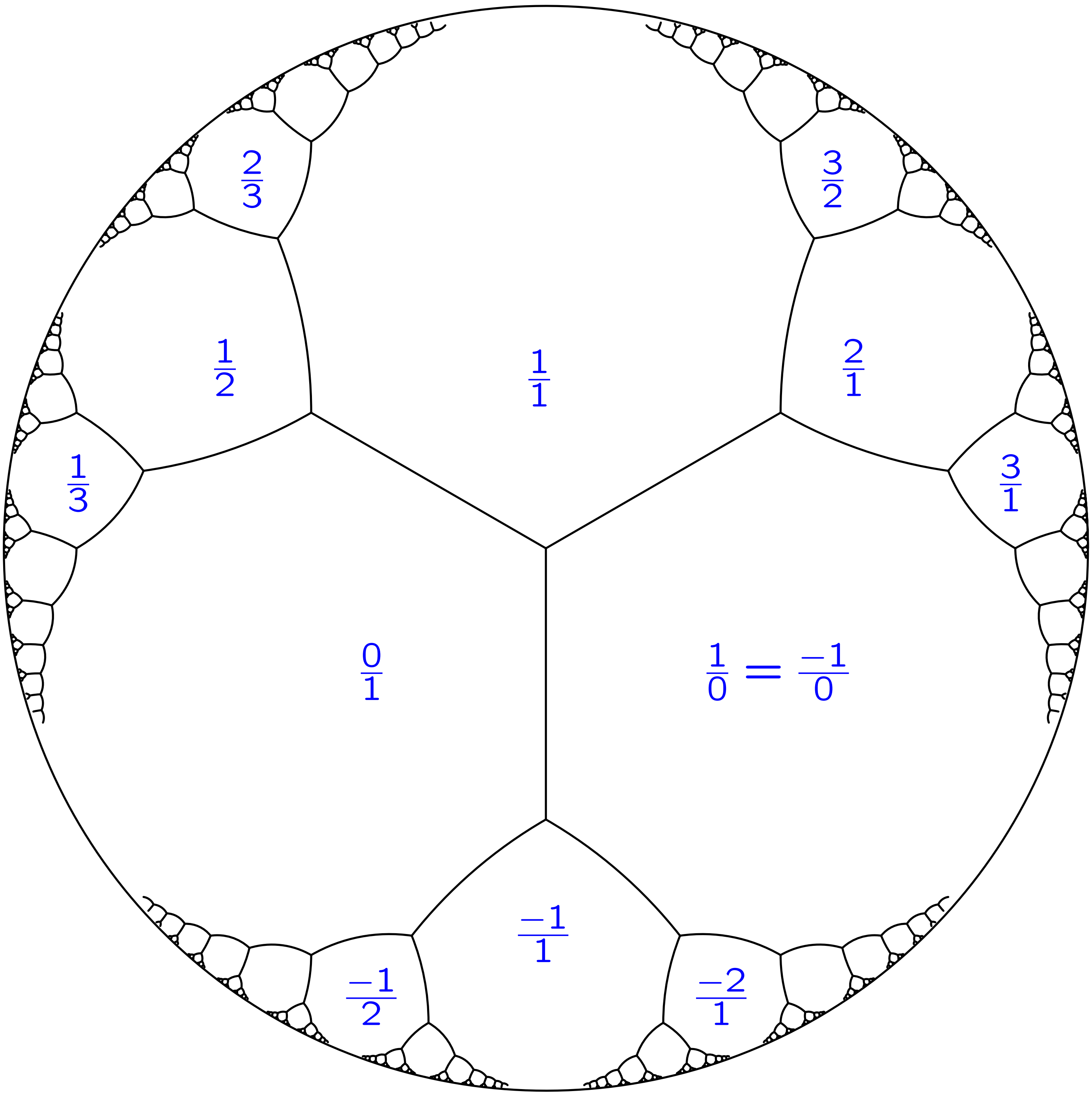
Slopes of essential s.c.c.'s in $T_{1,1}$

Relative to an ordered generating pair, the set SCC of all the essential s.c.c.'s in $T_{1,1}$ can be parametrized by their slopes in $\mathbb{Q}P^1 = \mathbb{Q} \cup \{\infty\}$.

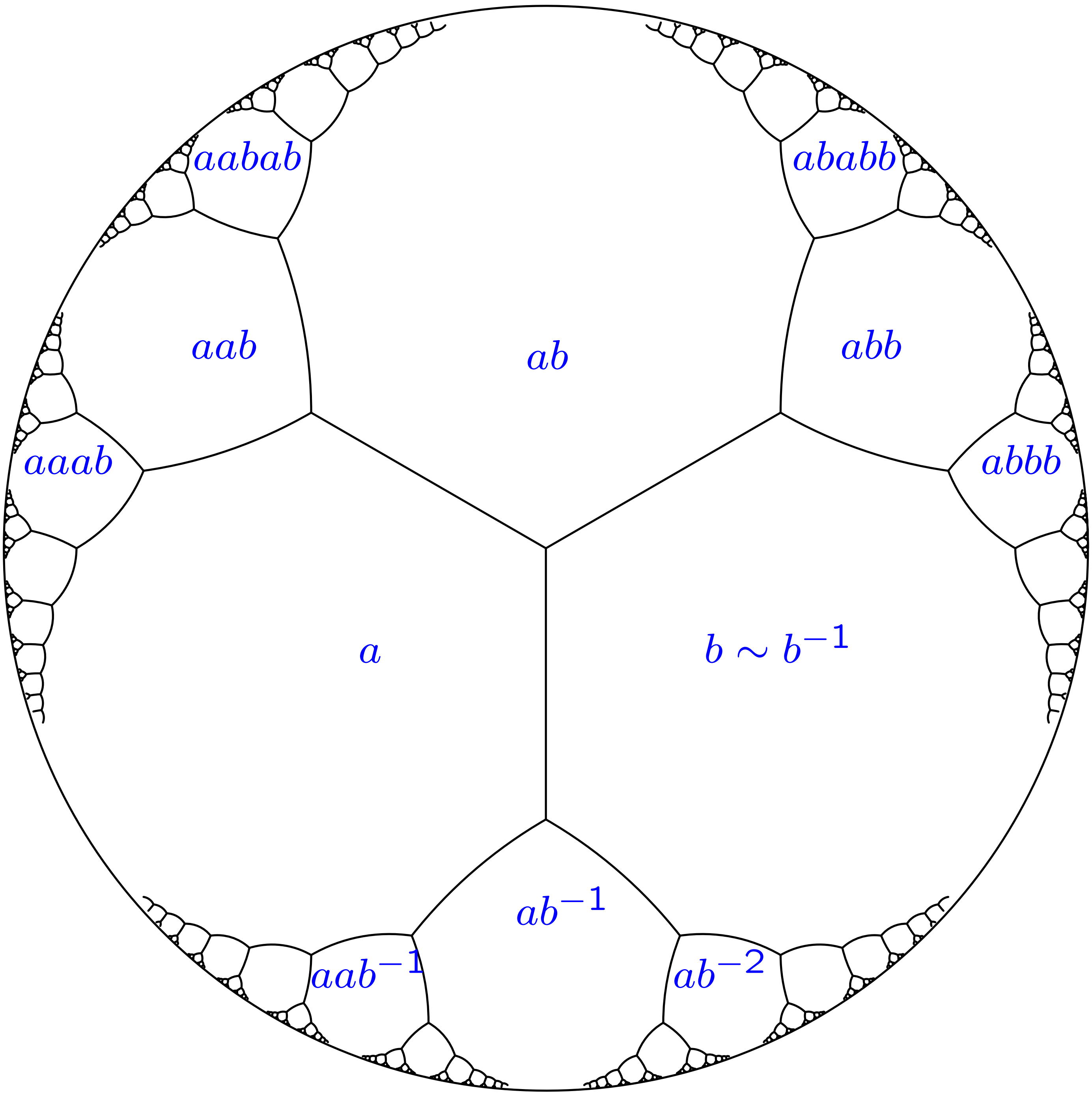
Hyperbolic plane $\mathbf{H}^2$ and 3-space $\mathbf{H}^3$

$$\mathrm{Isom}^+(\mathbf{H}^2) \equiv \mathrm{PSL}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R})/\{\pm I\}.$$

$$\mathrm{Isom}^+(\mathbf{H}^3) \equiv \mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm I\}.$$



Farey Tree of $\mathbb{Q}P^1$



Essential s.s.c.'s on $T_{1,1}$

For $A, B \in \mathsf{SL}(2,\mathbb{C})$,

$$\mathrm{tr}(A) = \mathrm{tr}(A^{-1}); \quad B + B^{-1} = \mathrm{tr}(B)\, I.$$

Fricke Trace Identities:

$$\begin{aligned} \mathrm{tr}(A)\,\mathrm{tr}(B) &= \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}); \\ \mathrm{tr}(ABA^{-1}B^{-1}) &= \mathrm{tr}^2(A) + \mathrm{tr}^2(B) + \mathrm{tr}^2(AB) \\ &\quad - \mathrm{tr}(A)\,\mathrm{tr}(B)\,\mathrm{tr}(AB) - 2. \end{aligned}$$

Write

$$\begin{aligned} x &= \mathrm{tr}(A), \\ y &= \mathrm{tr}(B), \\ z &= \mathrm{tr}(AB), \\ w &= \mathrm{tr}(AB^{-1}), \\ \kappa &= \mathrm{tr}(ABA^{-1}B^{-1}). \end{aligned}$$

Fricke Trace Identities:

$$\begin{aligned} z + w &= xy, \\ \kappa &= x^2 + y^2 + z^2 - xyz - 2. \end{aligned}$$

Note. $\kappa = x^2 + y^2 + w^2 - xyw - 2.$

Hyperbolic Structures on $\mathsf{T}_{1,1}$

A hyperbolic structure on $\mathsf{T}_{1,1}$ gives rise to a conjugacy class of representations

$$[\rho]:\ \pi_1(\mathsf{T}_{1,1}) \to \mathsf{PSL}(2,\mathbb{R}).$$

By taking a lift to $\mathsf{SL}(2,\mathbb{R})$, we may assume

$$\rho:\ \pi_1(\mathsf{T}_{1,1}) \to \mathsf{SL}(2,\mathbb{R}).$$

For a generating pair $\{a,\, b\}$, write

$$A = \rho(a),\ B = \rho(b) \in \mathsf{SL}(2,\mathbb{R}),$$

and, as above,

$$\begin{aligned} x &= \mathsf{tr}(A), \\ y &= \mathsf{tr}(B), \\ z &= \mathsf{tr}(AB); \\ \kappa &= \mathsf{tr}(ABA^{-1}B^{-1}). \end{aligned}$$

Then

$$\kappa = x^2 + y^2 + z^2 - xyz - 2.$$

Then $\mathsf{T}_{1,1}$ is a **hyperbolic torus** with

(1) one boundary geodesic: $\kappa \in (-\infty, -2)$;
(2) one cusp: $\kappa = -2$;
(3) one cone point: $\kappa \in (-2, 2)$.

In cases (1) and (3), we have

$$\kappa = -2\cosh\frac{l}{2} \quad \text{and} \quad \kappa = -2\cos\frac{\theta}{2},$$

where $l > 0$ and $\theta \in (0, 2\pi)$ are respectively the boundary geodesic length and cone angle.

Note. In all the 3 cases above,

$$[\rho(a)], [\rho(b)] \in \mathsf{PSL}(2, \mathbb{R})$$

are hyperbolic elements with crossing axes.

Note also that $\pi_1(\mathsf{T}_{1,1}) \cong \pi_1(\mathsf{S}_{0,3})$ where $\mathsf{S}_{0,3}$ is a thrice-punctured sphere.

Certain $\mathsf{SL}(2, \mathbb{R})$ characters of $\mathsf{T}_{1,1}$ correspond to hyperbolic structures on $\mathsf{S}_{0,3}$.

Some other $\mathsf{SL}(2,\mathbb{R})$ characters of $\mathsf{T}_{1,1}$ do **not** give any hyperbolic structures on $\mathsf{T}_{1,1}$ or $\mathsf{S}_{0,3}$.

Notation. Let $\mathfrak{X}$ be the set of all $\mathsf{SL}(2,\mathbb{C})$ characters of $\mathsf{T}_{1,1}$. For $\kappa \in \mathbb{C}$, let

$$\mathfrak{X}_\kappa := \{\, \rho \in \mathfrak{X} \mid \mathrm{tr}(ABA^{-1}B^{-1}) = \kappa \,\}.$$

An $\mathsf{SL}(2,\mathbb{C})$ character of $\mathsf{T}_{1,1}$ is **reducible** iff $\kappa = \mathrm{tr}(ABA^{-1}B^{-1}) = 2$. Thus,

$$\mathfrak{X}_{\mathbf{red}} = \mathfrak{X}_2.$$

We are interested in the set $\mathfrak{X}_{\mathbf{irr}}$ of irreducible $\mathsf{SL}(2,\mathbb{C})$ characters of $\mathsf{T}_{1,1}$

$$\mathfrak{X}_{\mathbf{irr}} := \mathfrak{X} \backslash \mathfrak{X}_{\mathbf{red}} = \mathfrak{X} \backslash \mathfrak{X}_2.$$

Write $(\mathbb{C}^3)_\kappa := \{\, x^2 + y^2 + z^2 - xyz - 2 = \kappa \,\}$.

Parametrization: $\mathfrak{X}_{\mathbf{irr}} \leftrightsquigarrow \mathbb{C}^3 \backslash (\mathbb{C}^3)_2$

Fix a generating pair $\{a, b\}$ of $\mathsf{T}_{1,1}$.

Given $\rho \in \mathfrak{X}$, we obtain $(x, y, z) \in \mathbb{C}^3$ where

$$x = \mathrm{tr}\,\rho(a), \quad y = \mathrm{tr}\,\rho(b), \quad z = \mathrm{tr}\,\rho(ab).$$

Conversely, given $(x, y, z) \in \mathbb{C}^3$ such that

$$\kappa := x^2 + y^2 + z^2 - xyz - 2 \neq 2,$$

there exist $A, B \in \mathrm{SL}(2, \mathbb{C})$, which are unique up to simultaneous conjugation, such that

$$x = \mathrm{tr}(A), \quad y = \mathrm{tr}(B), \quad z = \mathrm{tr}(AB).$$

— Actually, W. M. Goldman gives

$$A = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \zeta^{-1} \\ -\zeta & y \end{pmatrix},$$

where $\zeta \in \mathbf{C}$ satisfies $\zeta + \zeta^{-1} = z$.

Action of Mapping Class Group on $\mathfrak{X}$

Note that mapping class group MCG of $\mathrm{T}_{1,1}$ is isomorphic to $\mathrm{GL}(2, \mathbb{Z})$.

$\mathrm{GL}(2,\mathbb{Z})$ **acts on** $\mathfrak{X}_\kappa$ **or** $(\mathbb{C}^3)_\kappa$ as follows:

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}: \qquad (x,\, y,\, z) \longmapsto (x,\, xy - z,\, y)$$

corresponding to a left Dehn twist about a;

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}: \qquad (x,\, y,\, z) \longmapsto (xy - z,\, y,\, x)$$

corresponding to a right Dehn twist about b;

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}: \qquad (x,\, y,\, z) \longmapsto (x,\, y,\, xy - z)$$

corresponding to a reflection in a or b;

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}: \qquad (x,\, y,\, z) \longmapsto (y,\, x,\, xy - z)$$

corresponding to a reflection in ab or ab^{-1}.

Note: $\kappa = x^2 + y^2 + z^2 - xyz - 2$ is preserved under the action of $\mathrm{MCG} \equiv \mathrm{GL}(2,\mathbb{Z})$.

Definition. A character $\rho \in \mathfrak{X}$ is called a **Bowditch character** if
(1) $\operatorname{tr}\rho(\gamma) \notin [-2, 2]$ for all $\gamma \in$ SCC;
(2) $|\operatorname{tr}\rho(\gamma)| \leq 2$ for finitely many $\gamma \in$ SCC.

Proposition. A reducible character (in $\mathfrak{X}_2$) is never a Bowditch character.

Theorem. (Tan-Wong-Z, 2008)
(1) The Bowditch subset $\mathcal{B} \subset \mathfrak{X}$ is open.
(2) MCG acts on $\mathcal{B}$ discontinuously.

— The boundary of $\mathcal{B}$ in $\mathfrak{X}$ should be fractal.

Question (Minsky)

— Is $\mathcal{B} \subset \mathfrak{X}$ the **largest** such open subset?

(Generalized) Markoff Numbers

Markoff numbers = 1/3 times the positive integers in the MCC orbit of triple $(3, 3, 3)$:

1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597, 2897, 4181, 5741, 6466, 7561, 9077, 10946, 14701, 28657, 33461, 37666, 43261, 51641, 62210, 75025, 96557, 135137, 195025, 196418, 294685, 426389, 499393, 514229, 646018, 925765, $\cdots$

Markoff numbers were first studied by Markoff (Markov) in late 1870s in his work on minimum of indefinite, binary quadratic forms.

Relations with simple closed geodesics in the modular torus were later observed in 1950s.

Markoff numbers $m(r)$ can be indexed by their slopes $r \in \mathbb{Q} \cap [0, 1]$ as follows:

$$m(0/1) = 1,\ \ m(1/1) = 2,\ \ m(1/2) = 5,\ \ ...$$

Uniqueness Conjecture (1923). — Markoff numbers of different slopes are different.

Theorem (...) A Markoff number is unique if it is a 'small' multiple of a prime power.

Theorem (Z, 2007). A Markoff number m is unique if either $3m-2$ or $3m+2$ is k times a prime power, with $k \in \{1, 4, 8\}$.

Theorem (Z, 2007). Every **even** Markoff number m is $\equiv 2 \bmod 32$.

The list: 2, 34, 194, 610, 6466, 10946,
37666, 62210, 196418, 646018, $\cdots$

Numerical Results (Wu-Z). The first **100 million** Markoff numbers are unique.

Main Result

Theorem (Ji-Z) Let $\Lambda \subset \mathbf{C}$ be a discrete subring, and $\mu \in \Lambda$, $\mu \neq 4$. Then there exist only finitely many Bowditch characters in $(\Lambda^3)_\mu$.

Corollary of Main Result

Theorem (Ji-Z) For every integer $\mu \neq 4$, there exist only finitely many orbits of triples (x, y, z) of integers satisfying

$$x^2 + y^2 + z^2 - xyz = \mu.$$

Examples:

$\mu = 0$: $(0,0,0)$, $(3,3,3)$.

$\mu = -50$: $(3,7,9)$, $(4,5,7)$, $(5,5,5)$.

$\mu = -100$: $(3,10,11)$, $(5,5,10)$, $(5,6,7)$.

$\mu = 20$: $(0,-2,-4)$, $(-1,-2,-3)$, $(-2,-2,-2)$.

$\mu = 53$: $(0,-2,-7)$, $(-1,-2,-6)$,
$(-2,-2,-5)$, $(-2,-3,-4)$.

$\mu = 4$: $(n,n,2)$, $n = 0,1,2,\cdots$ (∞ many)

Thank You !