Index polynomial invariants of virtual knots and twisted knots.

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1. twisted link
2. Moves of Twisted links
3. Index Polynomial
4. An alternative definition of Index polynomial
5. Index polynomial of a twisted link
6. Refinement of Index polynomial

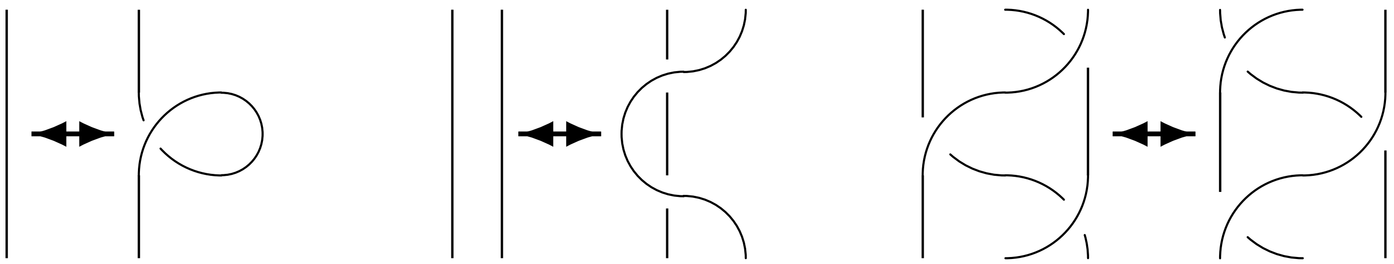
Twisted link diagram and virtual link diagram

D : twisted link diagram $\Leftrightarrow$ D : a link diagram whose double points are given the informations over/under or virtual possibly with some bars on arcs

D : virtual link diagram $\Leftrightarrow$ D : a twisted link diagram without bars on arcs

Generalized Reidemeister moves

Reidemeister moves



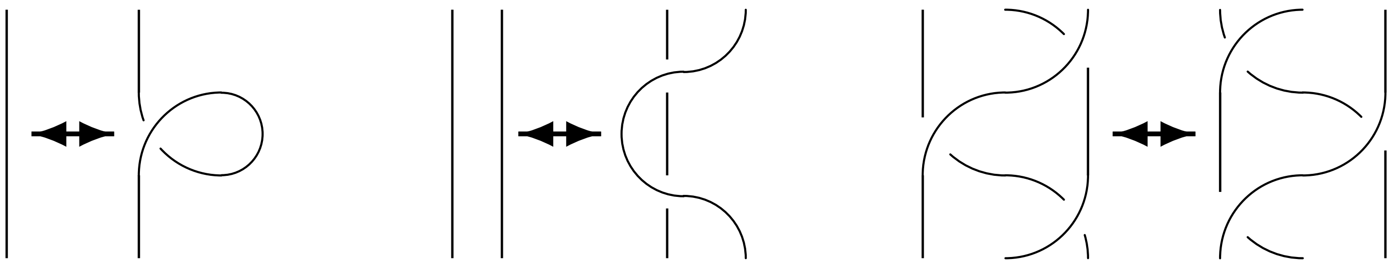
I II III

Virtual Reidemeister moves

I II III IV

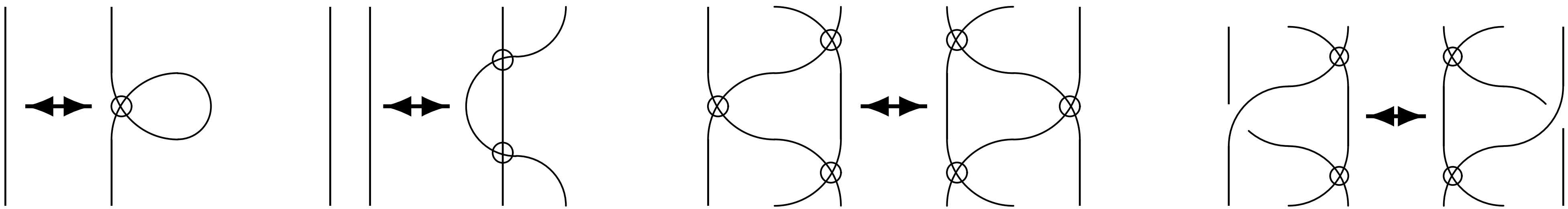
Generalized Reidemeister moves

Reidemeister moves



I II III

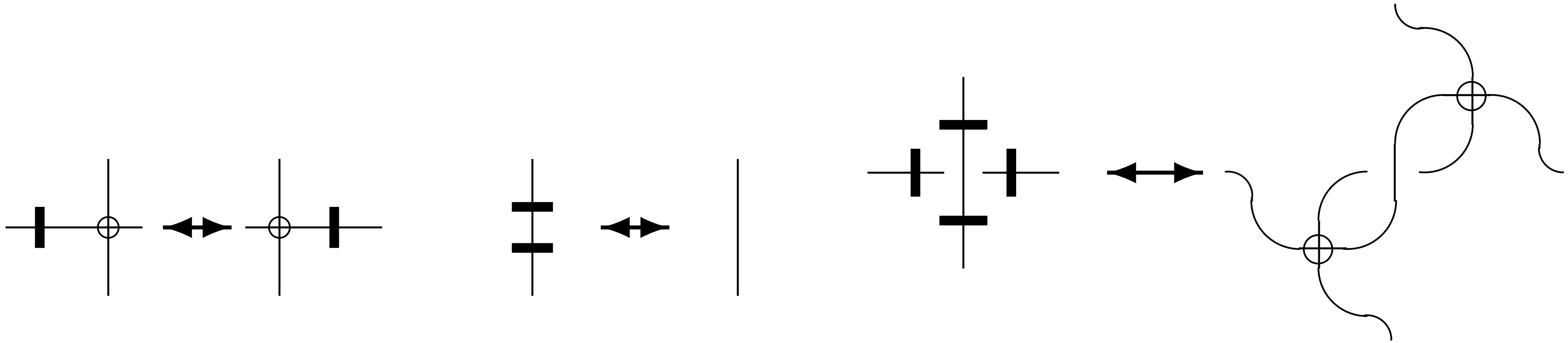
Virtual Reidemeister moves



I II III IV

Extended Reidemeister moves

Generalized Reidemeister moves +



I II III

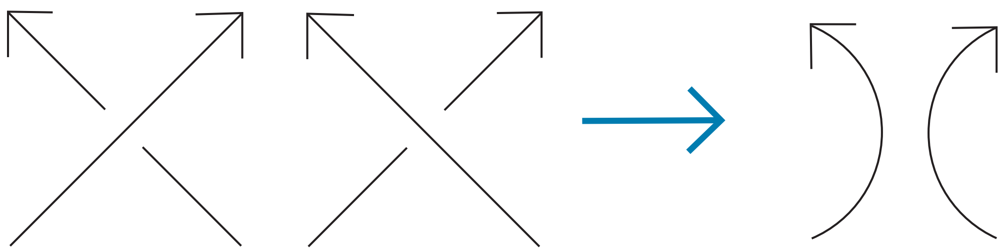
Twisted link and virtual link

A twisted link is the equivalence class of a twisted link diagram under Reidemeister moves I, II, III, virtual Reidemeister moves I, II, III, IV and twisted Reidemeister moves I, II, III.
A virtual link is the equivalence class of a virtual link diagram under Reidemeister moves I, II, III and virtual Reidemeister moves I, II, III, IV.

Index diagram

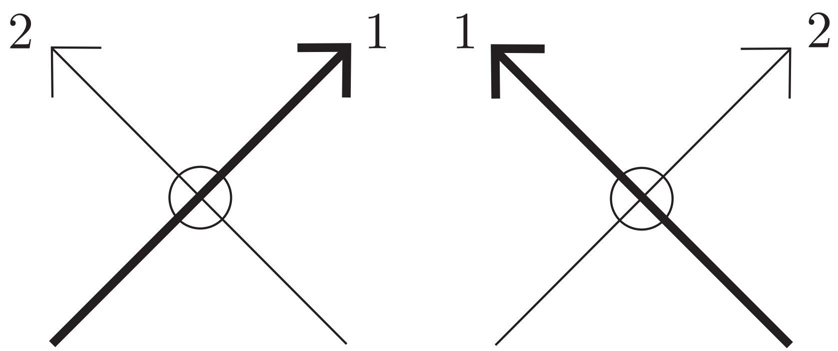
D : a virtual link diagram
c : a real crossing of D
The index diagram of a real crossing c, D_c is defined as follows;

- If c is the real crossing of the distinct components of D, say d_1 and d_2, $D_c = d_1 \cup d_2$
- If c is the real crossing of a component d of a virtual link diagram D, D_c is two-component link diagram obtained from d by smoothing at c



Index

We label each component of D_c with $(1,2)$, then the index of a virtual crossing v of D_c is defined as follows:



$ind(v) = 1$ $\quad$ $ind(v) = -1$

The virtual intersection index $i(c)$ of c is defined by

$$i(c) = \sum_{v \in 1 \cap 2} ind(v)$$

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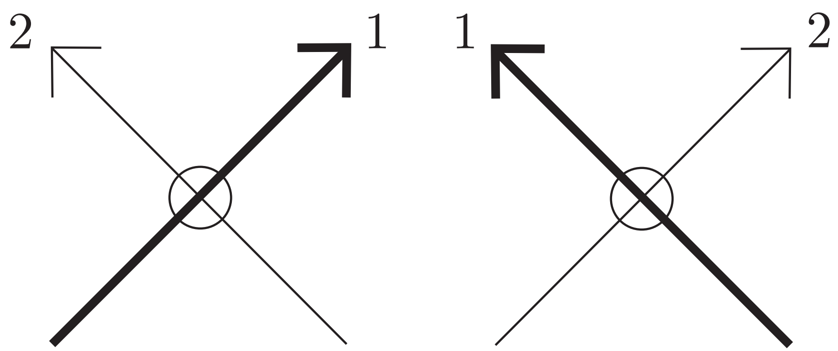

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Definition of Index polynomial

Index Polynomial

$Q_D(t) = \sum_c sgn(c)(t^{|i(c)|} - 1)$, where c runs over all real crossings of D and $sgn(c)$ is the sign of a real crossing c.

Theorem (Y. H. Im, K. Lee, S. Y. Lee)

$Q_D(t)$ *is an invariant of a virtual link.*

Corollary (Y. H. Im, K. Lee, S. Y. Lee)

For a virtual link L*, the virtual crossing number of* L *is greater than or equal to the maximal degree of* t *in* $Q_D(t)$*, where* D *is a virtual link diagram of* L*.*

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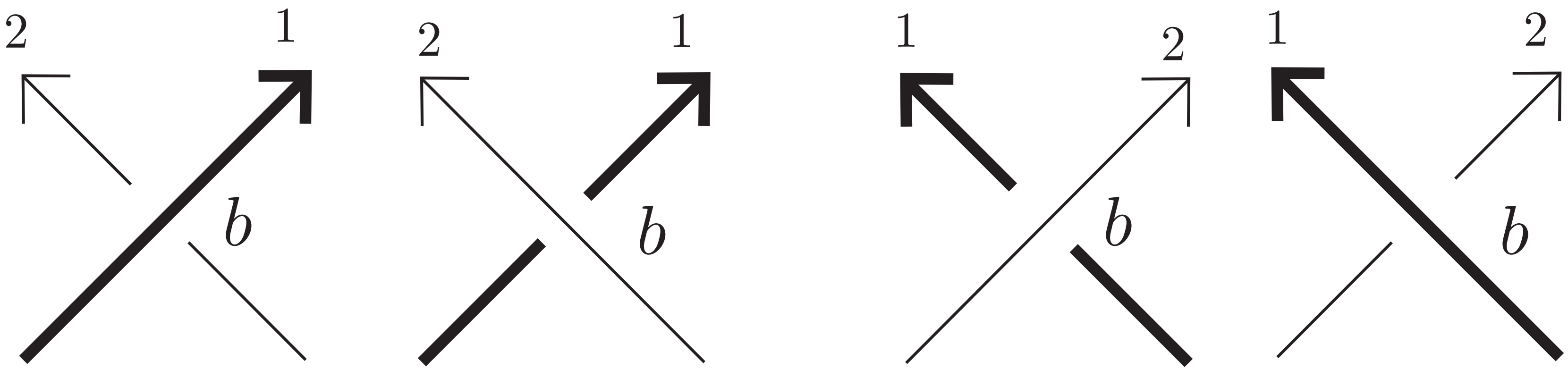
For a virtual link L, the virtual crossing number of L is greater than or equal to the maximal degree of t in $Q_D(t)$, where D is a virtual link diagram of L.

Index of a real crossing

D: a twisted link diagram
c : a real crossing of D
We label each component of Index diagram of D_c with $(1, 2)$.



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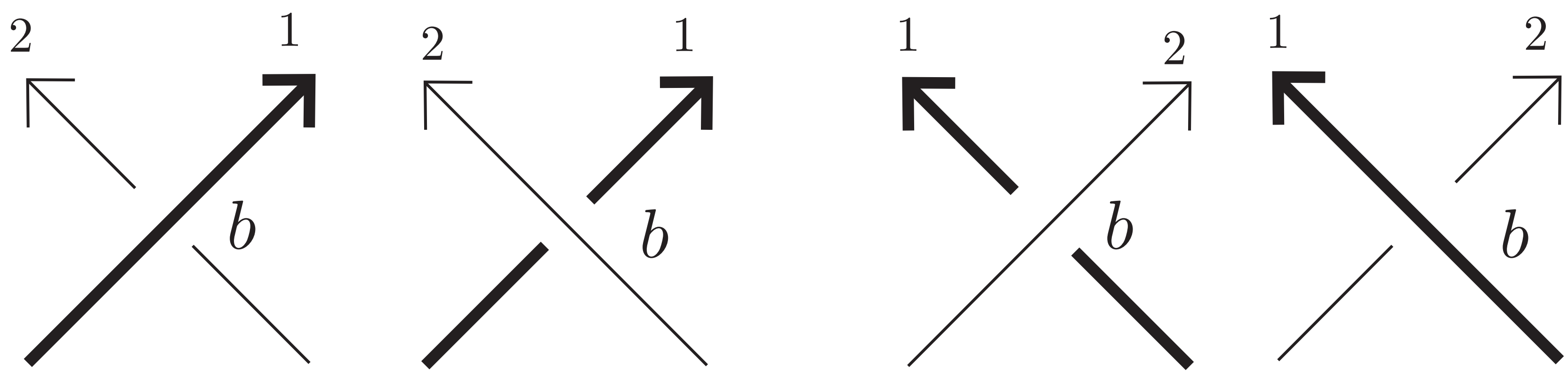

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An alternative definition

Since the algebraic intersection number of two components of D_c is zero, we have

$$i(c) + i_r(c) = \sum_{v \in 1 \cap 2} ind(v) + \sum_{b \in 1 \cap 2} ind_r(b) = 0.$$

So we see $|i(c)| = |i_r(c)|$

Theorem

$$Q_D(t) = \sum_c sgn(c)(t^{|i_r(c)|} - 1),$$

where c runs over all real crossing of D and $sgn(c)$ is the sign of a real crossing c.

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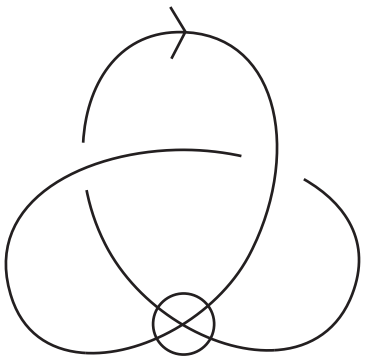
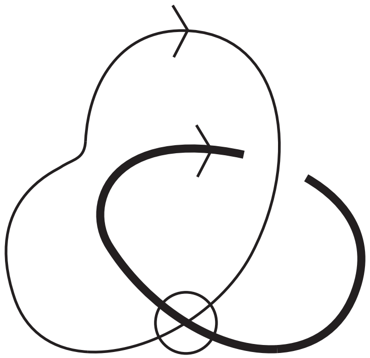
Corollary and Example

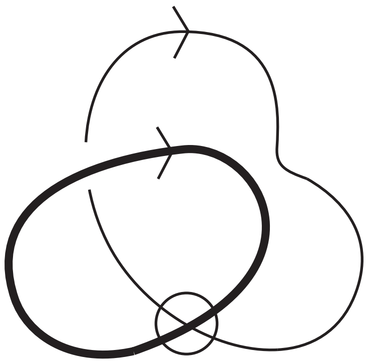
Corollary

For a virtual link L, the real crossing number of L is greater than or equal to the maximal degree of t in $Q_D(t)$, where D is a virtual link diagram of L.





(i)



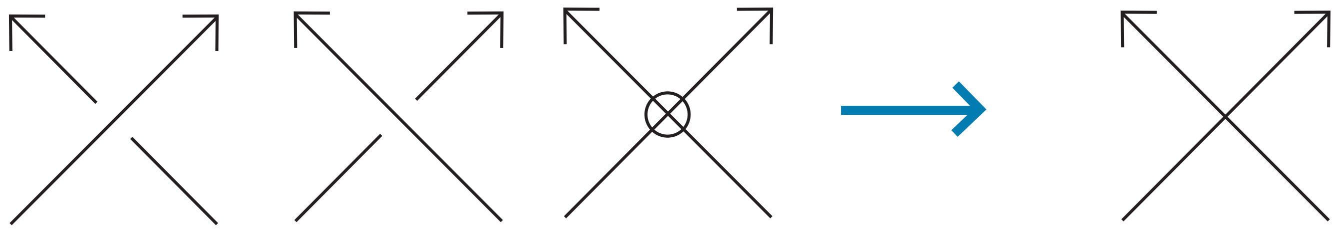
(ii)

	(i)	(ii)
i	1	-1
i_r	-1	1

$$Q_D = 2(t-1)$$

bar-edge

$\boldsymbol{p}$: {twisted link diagram } $\longrightarrow$ {immersed loops with some bar}



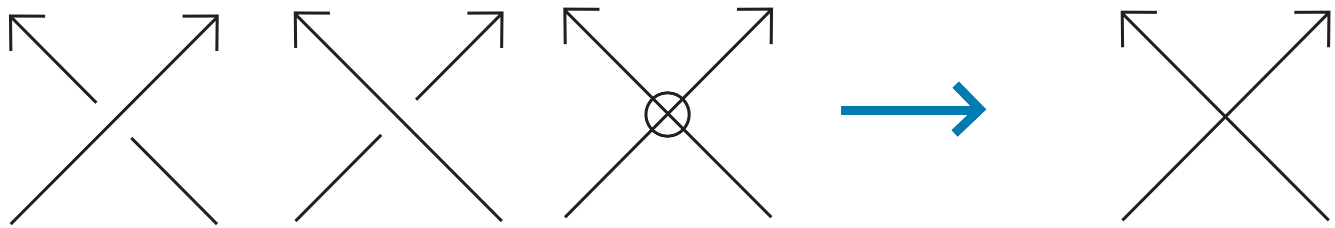
D : twisted link diagram

e: bar-edge of D $\Leftrightarrow$ Preimege of a segment of $p(D)$ between two bars

2¿baredge.eps

bar-edge

p: {twisted link diagram } $\longrightarrow$ {immersed loops with some bar}



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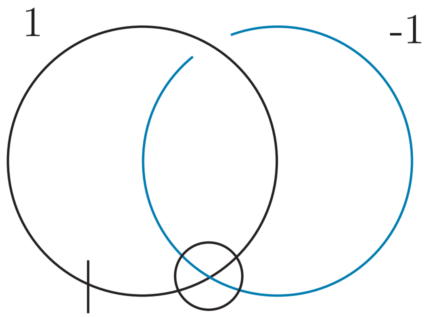
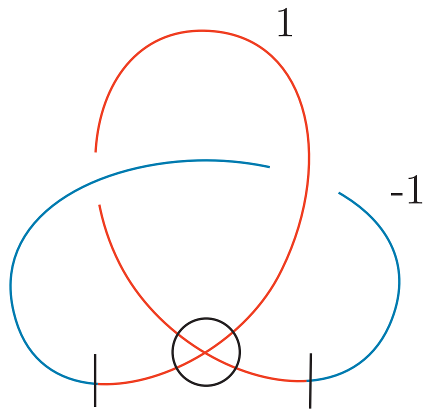
Weight map

D : twisted link diagram

A component of D is said to be even(or odd) if there are even (or odd) number of bars on it.



odd and even



even

$E(D)$: the set of bar-edge of D

σ : weight map of $D \Leftrightarrow$

σ : E(D) $\longrightarrow \{1, -1\}$

s.t. $\sigma(e) \neq \sigma(e')$ for $e, e' \in E(D)$ if e and e' contained in an even component and they are adjacent.

$\sigma(e) = \sigma(e')$ for $e, e' \in E(D)$ if e and e' contained in an odd component.

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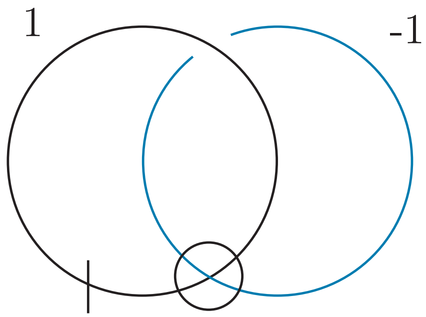

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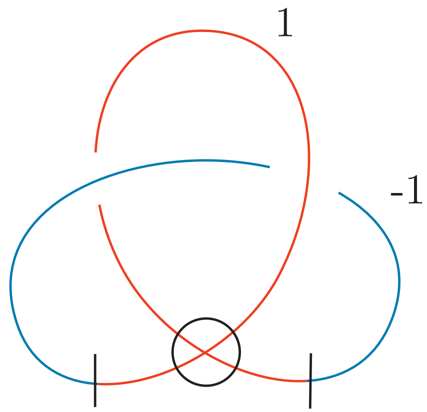

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Index of a twisted link

The index diagram of a real crossing c of a twisted link diagram D, D_c, is defined in the same manner as the index diagram for a virtual link diagram. For a weight map σ of D_c, we define the r-index of real crossing b, $ind_\sigma(b)$ as follows.

The r-index of D_c with respect to a weight map σ is given by

$$i_\sigma(c) = \sum_{b \in \mathcal{R}(D_c)} ind_\sigma(b),$$

where $\mathcal{R}(D_c)$ is a set of real crossings of D_c which is between the distinct components of D_c.

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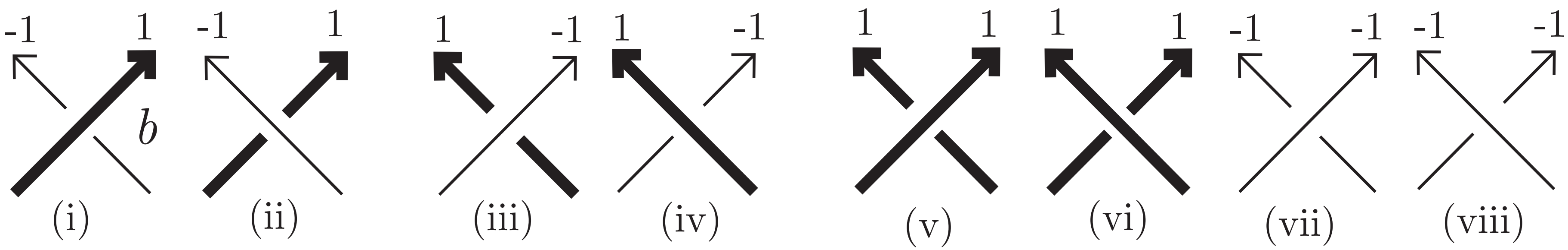

1 -1 0

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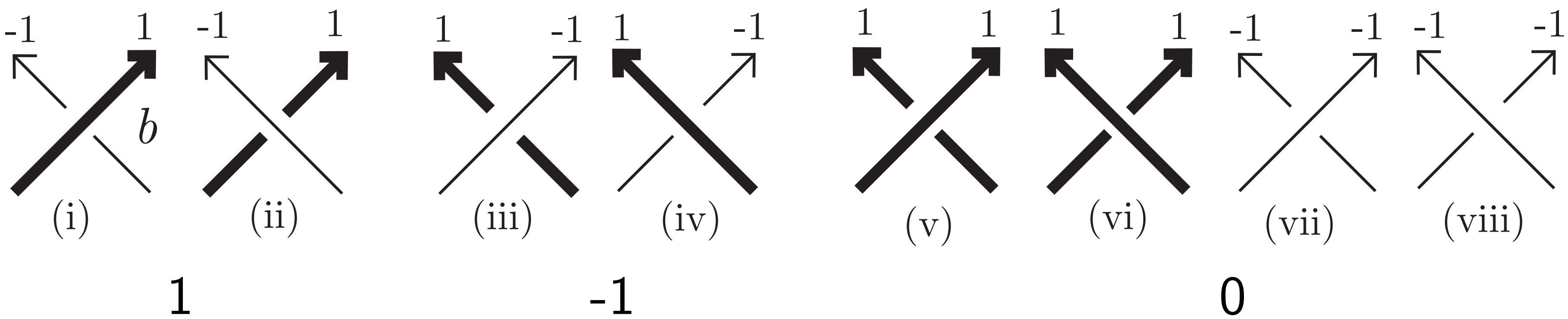

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Index polynomial of twisted link

The map g : {real crossings of D} $\longrightarrow \mathbb{Z}(t^{\pm 1}, s, u)$ is defined by

$$g(c) = \begin{cases} \sum_{\sigma}(t^{i_\sigma(c)} - 1) & \text{if two components of } D_c \text{ are even} \\ \sum_{\sigma}(s^{[i_\sigma(c)]} - 1) & \text{if two components of } D_c \text{ are odd} \\ u^{[lk(D_c)]} - 1 & \text{if one component of } D_c \text{ is even and the other is odd} \end{cases}$$

where $\sum$ runs over all weight maps of D_c, $[m]$ an integer in $\{0, 1\}$ congruent to m modulo 2 for an integer m,
$lk(D_c)$:the linking number of two components of D_c which is the sum of signs of all real crossings between the distinct components.

Index polynomial

Theorem

$\bar{Q}_D = \sum_c sgn(c)g(c)$ *is an invariant of a twisted link.*

Corollary

For a twisted link L*, the real crossing number of* L *is greater than or equal to the maximal degree of* t *in* $\bar{Q}_D$*, where* D *is a twisted link diagram of* L*.*

Corollary

For a twisted link L*, the real crossing number of* L *is greater than or equal to one forth of the constant term in* $\bar{Q}_D$*, where* D *is a twisted link diagram of* L*.*

Index polynomial

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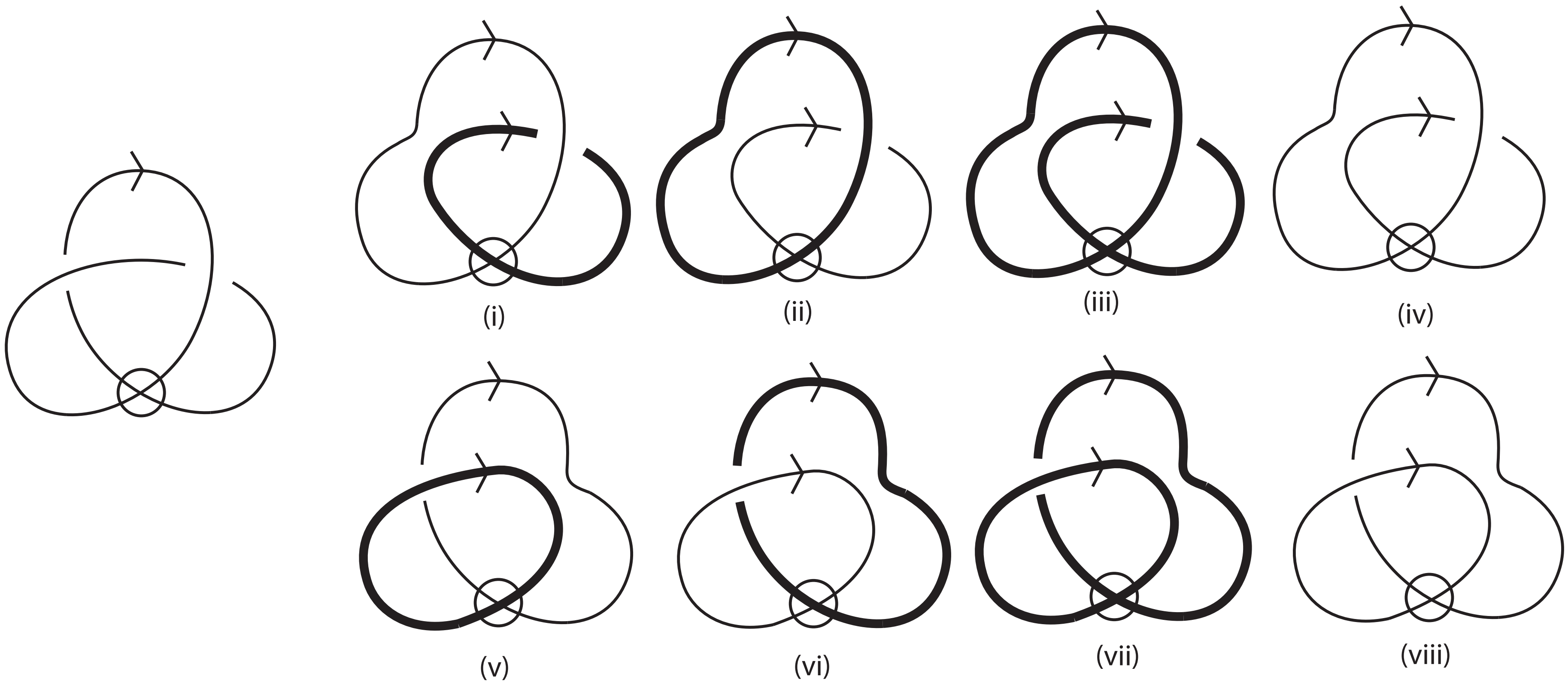
For a twisted link L, the real crossing number of L is greater than or equal to one forth of the constant term in $\bar{Q}_D$, where D is a twisted link diagram of L.

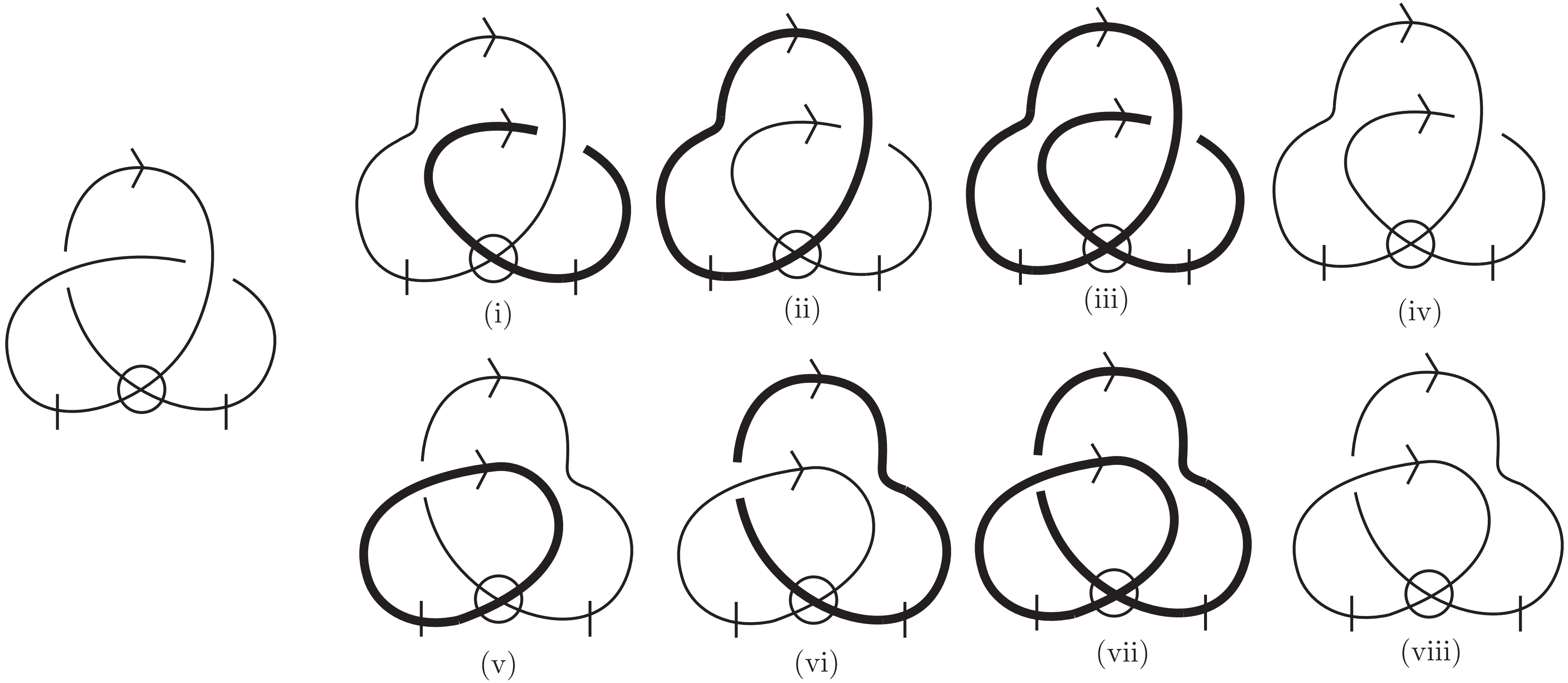
Example 1



σ	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
i_σ	-1	1	0	0	1	-1	0	0

$$\bar{Q}_D = 2(t + t^{-1} - 2)$$

Example 2

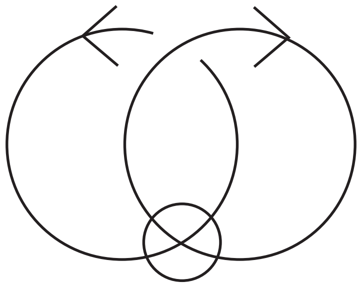


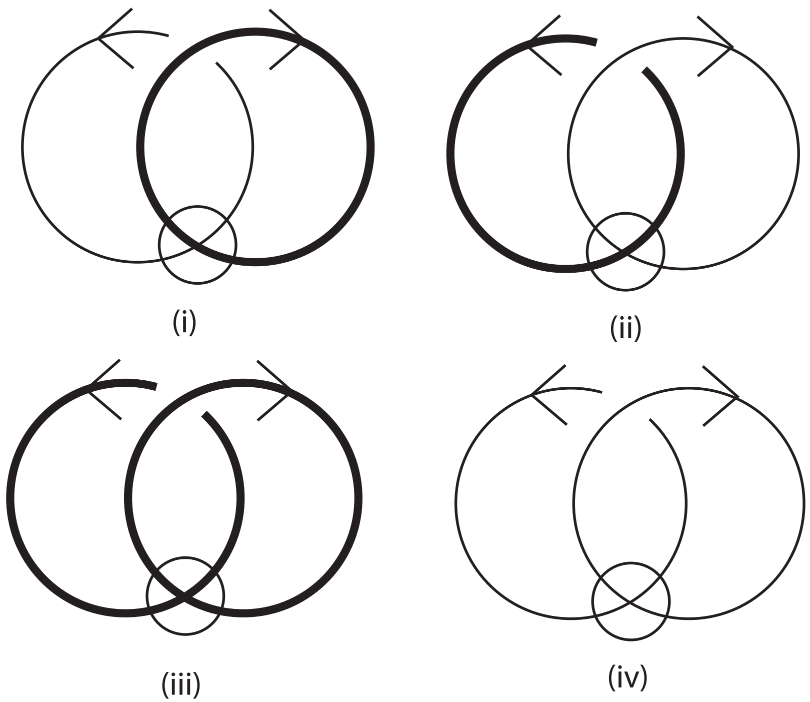
σ	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
i_σ	-1	1	0	0	1	-1	0	0

$\bar{Q}_D = 4(s-1)$

The twisted Jones polynomials of two diagrams in Examples 1 and 2 are $-A^{-4}(A^2 + A^{-2})(1 + A^{-2} + A^{-6})$.

Example 3

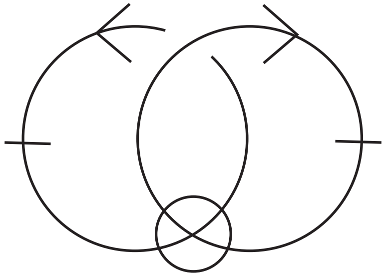


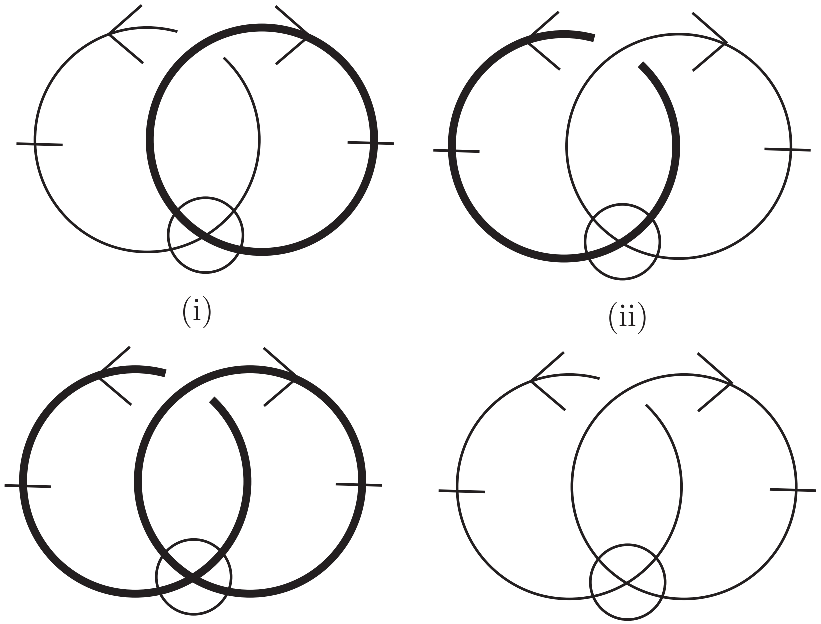


	(i)	(ii)	(iii)	(iv)
i_σ	1	-1	0	0

$$\bar{Q}_D = t + t^{-1} - 2$$

Example 4





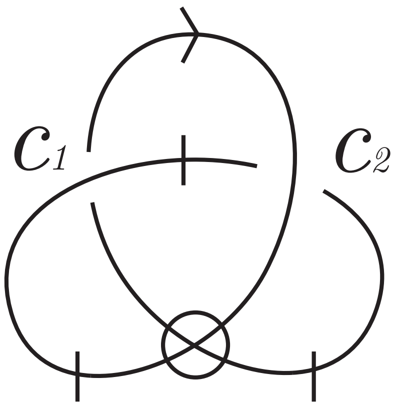
(i) (ii) (iii) (iv)

	(i)	(ii)	(iii)	(iv)
i_σ	1	-1	0	0

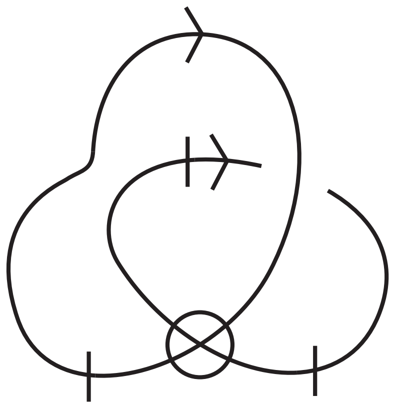
$$\bar{Q}_D = 2s - 2$$

The Twisted Jones polynomials of two diagrams in Examples 1 and 2 are $A^{-3}(A + A^{-1})(A^2 + A^{-2})$.

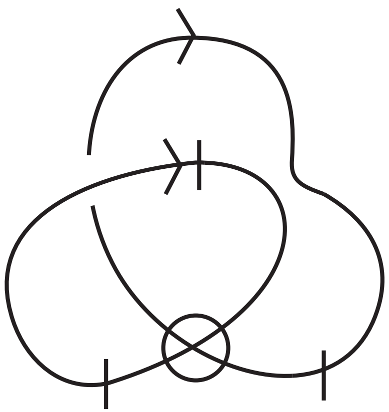
Example 5



(i)



(ii)

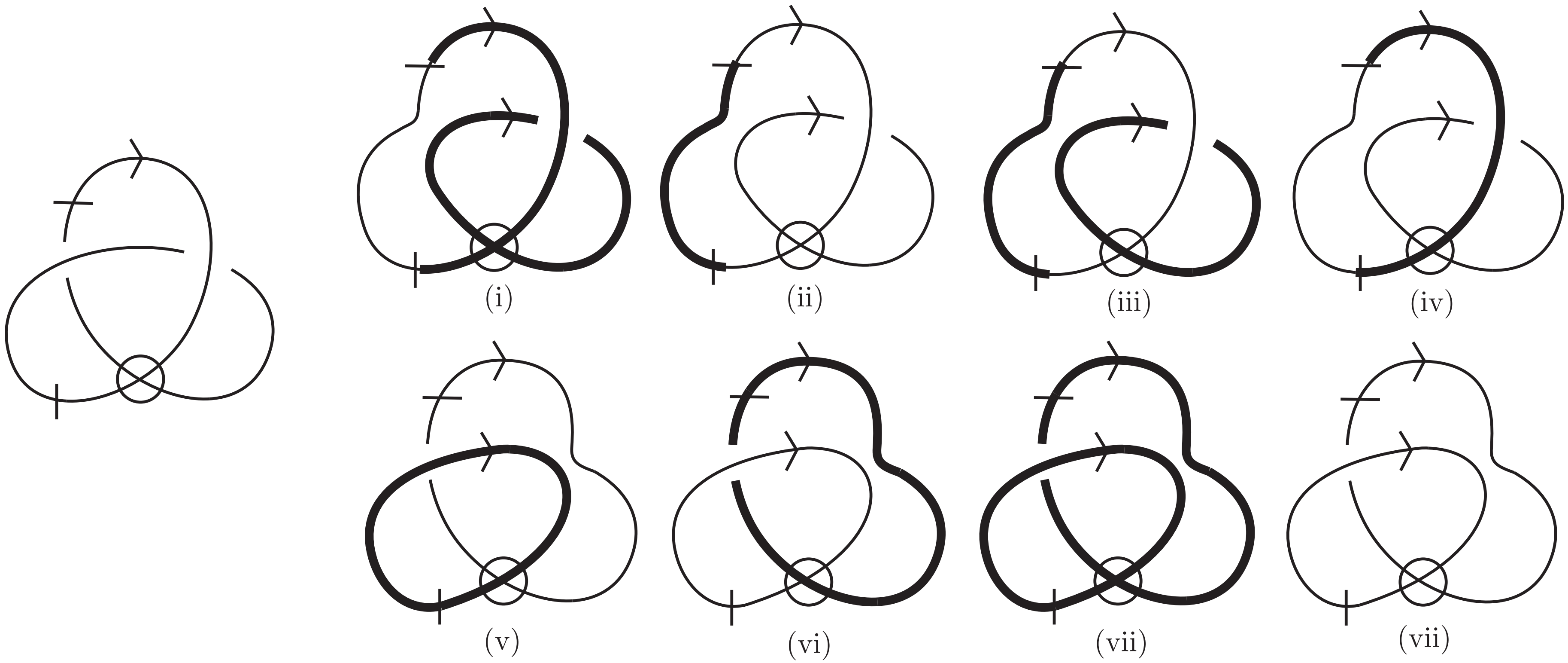


(iii)

	(i)	(ii)
lk	1	1

$$\bar{Q}_D = 2(u-1)$$

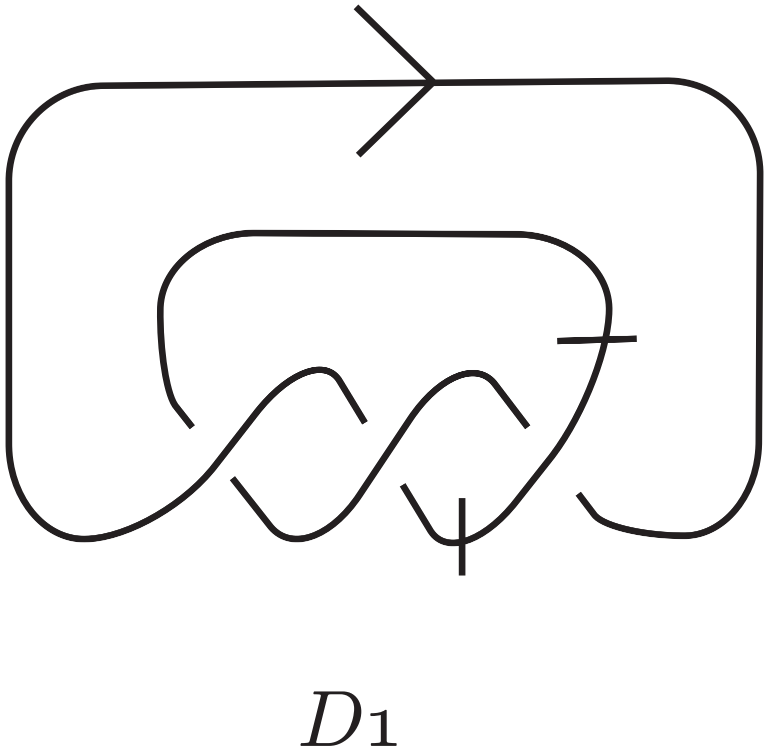
Example 6

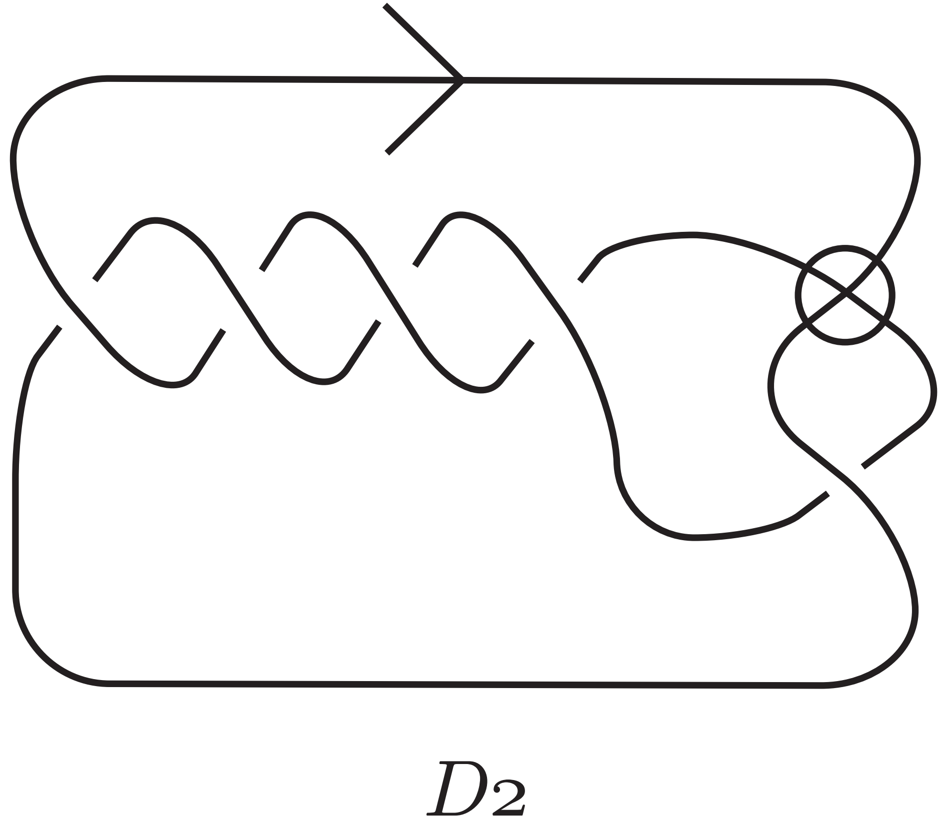


	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
i_σ	0	0	-1	1	1	-1	0	0

$$\bar{Q}_D = t + t^{-1} + 2s - 4$$

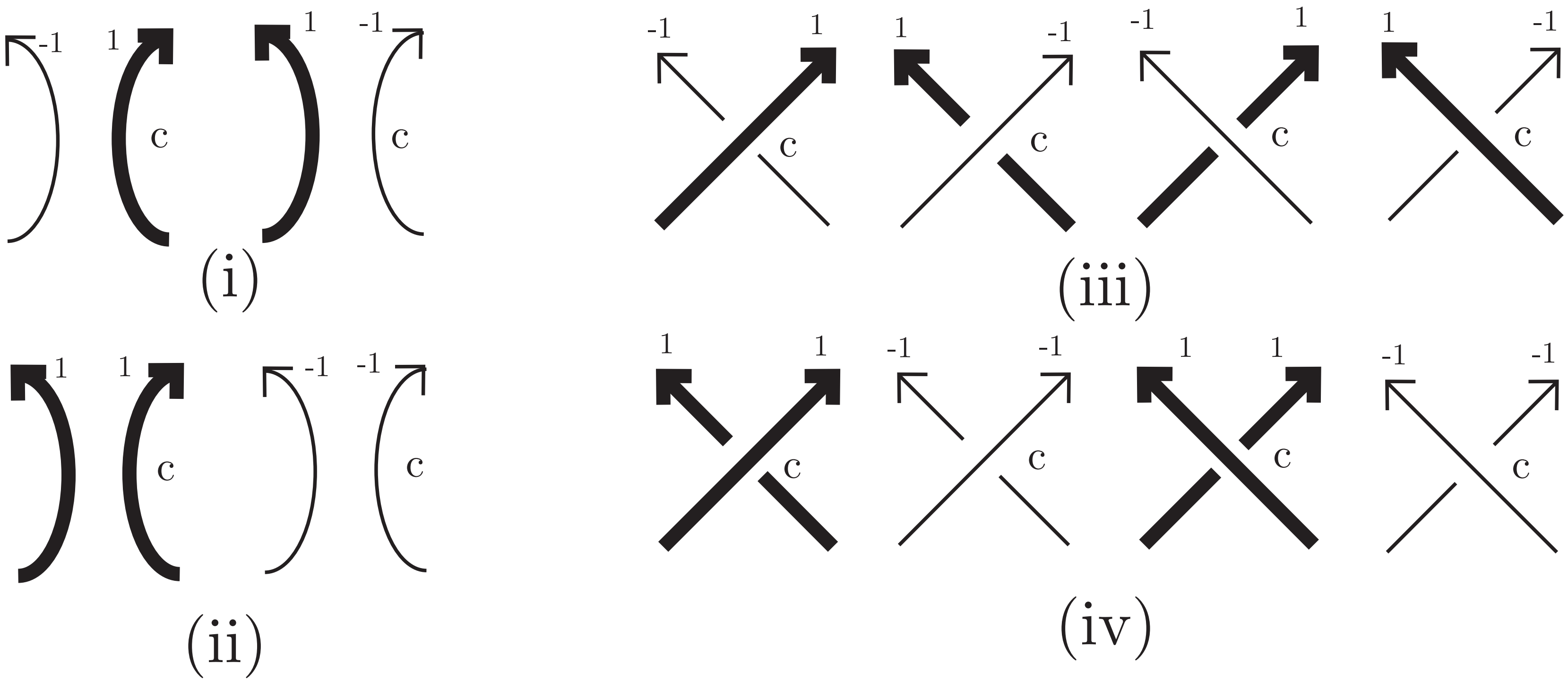
Example 7





$$\bar{Q}_{D_1} = \bar{Q}_{D_2} = -4(t + t^{-1} - 2)$$

Refinement of Index polynomial



Let D be a twisted link diagram and c a real crossing of D. We denote by $\mathcal{W}_1(c)$ the set of weight maps of D_c such that it looks like Figure (i) around c. Similarly, let $\mathcal{W}_2(c)$, $\mathcal{W}_3(c)$ and $\mathcal{W}_4(c)$ be the set of weight maps of D_c as in Figures (ii), (iii) and (iv), respectively. Every weight map of D_c belongs to one of $\mathcal{W}_1(c), \ldots, \mathcal{W}_4(c)$.

Refinement of Index polynomial

The map
$\tilde{g}$: {real crossings of D} $\longrightarrow \mathbb{Z}(t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}, t_4^{\pm 1}, s_1, s_2, s_3, s_4, u)$ is defined by

$$\tilde{g}(c) = \begin{cases} \sum_k \sum_{\sigma \in \mathcal{W}_k(c)} (t_k^{i_\sigma(c)} - 1) & \text{if two components of } D_c \text{ are even} \\ \sum_k \sum_{\sigma \in \mathcal{W}_k(c)} (s_k^{[i_\sigma(c)]} - 1) & \text{if two components of } D_c \text{ are odd} \\ u^{[lk(D_c)]} - 1 & \text{if one component of } D_c \text{ is even and the other is odd} \end{cases}$$

Theorem

$\widetilde{Q}_D = \sum_c sgn(c)\tilde{g}(c)$ *is an invariant of a twisted link.*

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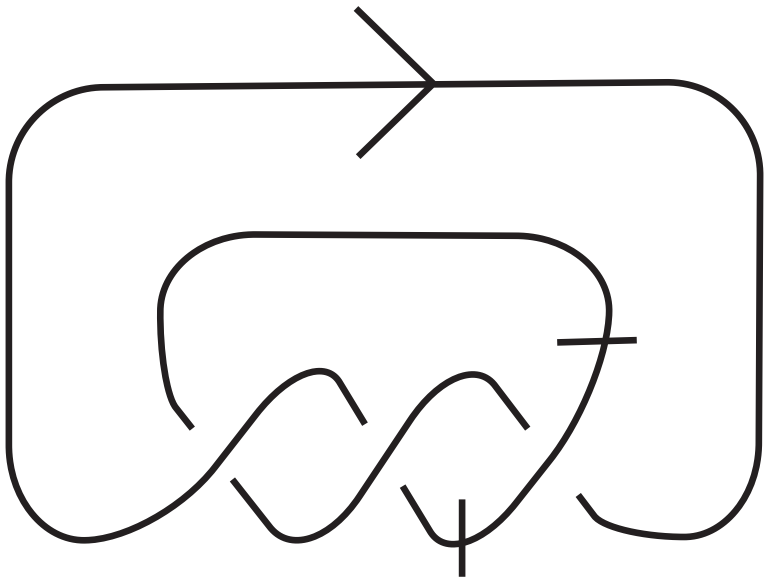
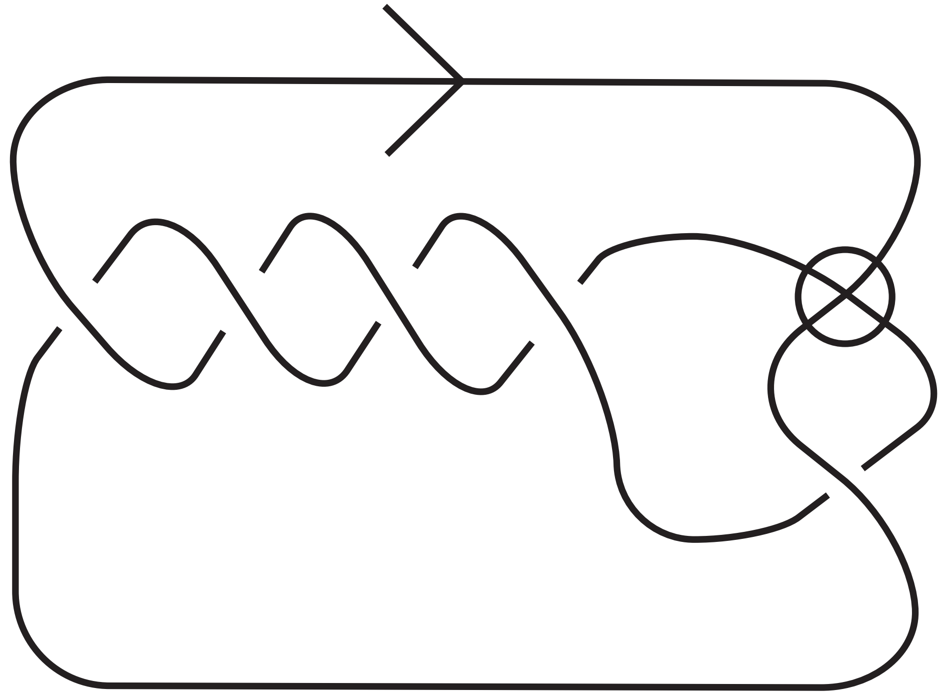
Example 7



$D1$

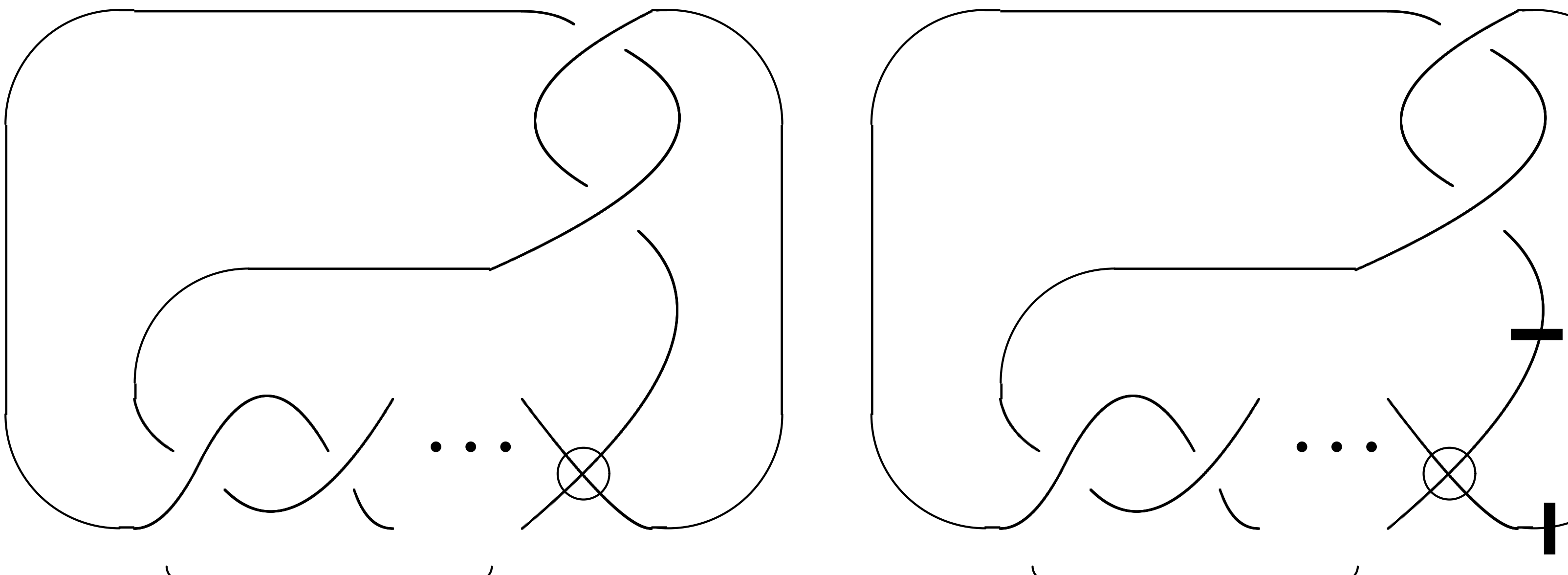


$D2$

$$\bar{Q}_{D_1} = -\bar{Q}_{D_2} = 4(t + t^{-1} - 2)$$

$$\widetilde{Q}_{D_1} = -2(t_1 + t_1^{-1} + t_2 + t_2^{-1} - 4), \ \widetilde{Q}_{D_2} = -4(t_1 + t_1^{-1} - 2)$$

Example 8



D_m D'_m

$$\bar{Q}_{D_m} = 2(-1)^m(t + t^{-1} - 2),\ \bar{Q}_{D'_m} = 4(-1)^m(s - 1)$$

The Twisted Jones polynomials of two diagrams D_m and D'_m are $-A^{-6}(A^4 + A^{-4}) - A^{-4m}(A^3 - A^{-3})(A + A^{-1})$ (or $-A^6(A^4 + A^{-4}) + A^{-4m+12}(A^3 - A^{-3})(A + A^{-1})$) if m is even (or odd).