Volume preserving moves on graphs and their applications

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Hyperbolic graph

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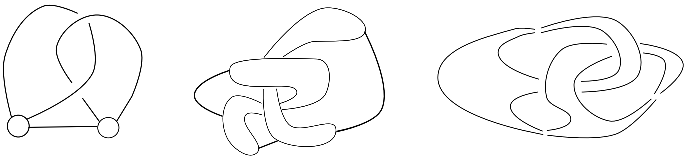
Definition

A spacial graph G is hyperbolic if N_G admits complete hyperbolic structure (with parabolic meridians) of finite volume with totally geodesic boundaries.

Example (Intuitive picture) of a hyperbolic graph.



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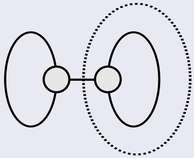

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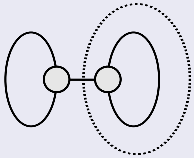

- 2-irreducible if every 2-sphere in M meeting G transversely in two points bounds a ball in M that intersects G in a single unknotted arc.

Theorem (Heard-Hodgson-Martelli-Petronio)

G: a trivalent spatial graph in M.
G is hyperbolic $\iff$

- *G is $(0, 1, 2)$-irreducible,*
- *N_G is homotopically atoroidal and is not a solid torus or the product of a torus with an interval, and*
- *G is not the trivial θ-graph in S^3.*

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Remark. This theorem is based on Thurston's uniformization theorem.

From now on, we only consider hyperbolic graphs in S^3.

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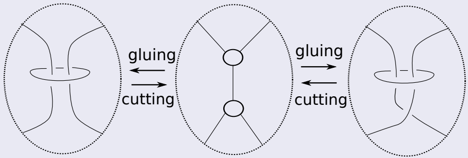
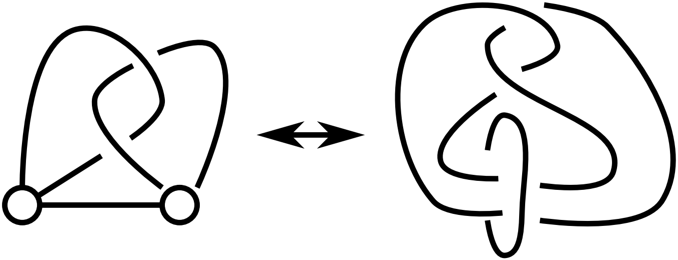
Lemma

The following moves on hyperbolic graphs in S^3 are volume preserving.



This lemma relates hyperbolic graphs with hyperbolic links.

Example.



Two complements have the same volume.

Theorem (Lackenby, Agol-Thurston)

L : hyperbolic link with a diagram D
$\mathrm{t}(D)$: twist number of D

$$\mathrm{Vol}(S^3 \setminus L) \leq 10v_3(\mathrm{t}(D) - 1)$$

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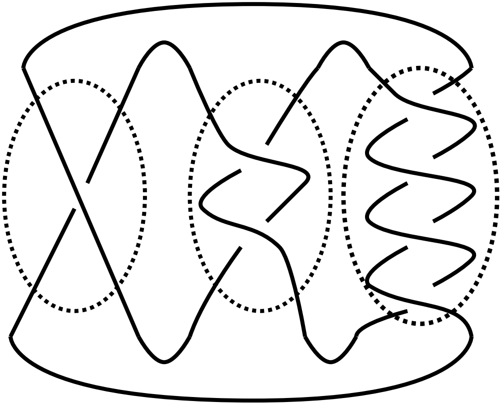
This upper bound is asymptotically sharp.

Definition

A twist is either a connected collection of bigon regions arranged in a row, which is maximal in the sense that it is not part of a longer row of bigons, or a single crossing adjacent to no bigon regions.



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The twist number of a diagram D is its number of twists.

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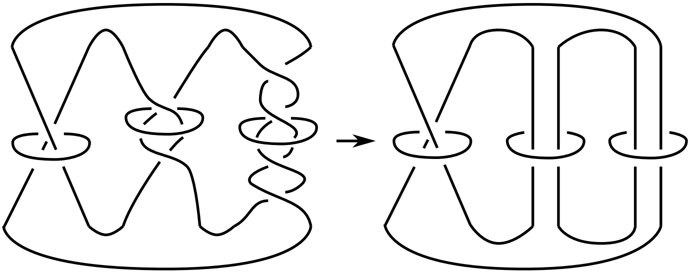
A fully augmented link is a link obtained by encircling each twist by a single unknoted component.



By Thurston's hyperbolic Dehn surgery theorem,

Fact

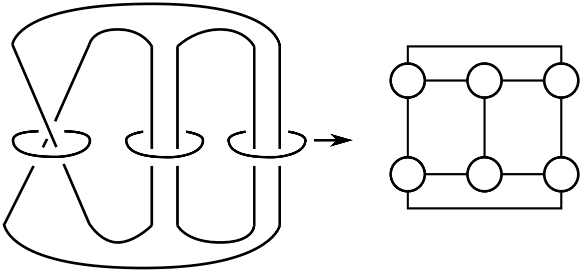
L : hyperbolic link with a diagram D,
F : fully augmented link obtained from D.
$\Rightarrow \mathrm{Vol}(S^3 \setminus F) > \mathrm{Vol}(S^3 \setminus L)$.

Proposition

F : Fully augmented hyperbolic link obtained from a diagram D
By cutting moves, we get a

- *simple*
- *planar*
- *trivalent*

graph with $2t(D)$ vertices.

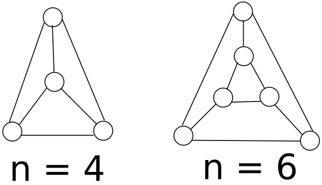


Since one can enumerate all simple planar trivalent graphs with n-vertices, Since one can enumerate all simple planar trivalent graphs with n-vertices,

Theorem

For given $n \in \mathbb{N}$, there is an algorithm to compute the best possible upper bound of the volume of links which have a diagram with n twists.

Example.
The unique simple planar trivalent graph with 4 (or 6) vertices.



Their volumes are $2v_8 \approx 7.327$ and $4v_8 \approx 14.65$ respectively.
$(10v_3(2-1) \approx 10.14$ and $10v_3(3-1) \approx 20.29)$

- The program called *plantri* (by Brinkmann and McKay) can enumerate all simple planar trivalent graphs with n vertices.
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Definition

- A trivalent graph is cyclically k-connected if it has no non-trivial t-cuts for $0 \leq t \leq k-1$

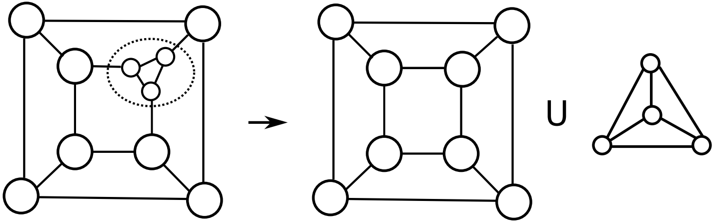
Remark. If a spatial graph G with a planar diagram has a 1 or 2-cut, then G cannot be hyperbolic.

Theorem (Adams)

Every essential thrice punctured sphere in a hyperbolic 3-manifold is isotopic to a totally geodesic one.

$\Rightarrow$ if a spatial graph G with a planar diagram has a non-trivial 3-cut, then it reduces to a case of fewer vertices.

Example.



$\omega(P) := \mathrm{Vol}(N_P)/n$ (P a hyperbolic graph with n vetices).
$U_m := \max\{\omega(Q) | Q$ trivalent hyperbolic planar graph with m vertices$\}$.

Proposition

If P has a non-trivial 3-cut, then for some $4 \leq k \leq n-2$, $\omega(P) \leq U_k$.

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Proposition

If P has a non-trivial 3-cut, then for some $4 \leq k \leq n-2$, $\omega(P) \leq U_k$.

By this proposition , if we find a cyclically 4-connected graph P_n with n vertices such that $\omega(P_n) \geq U_k$ for all $4 \leq k \leq n-2$, then we do not have to look at graphs with non-trivial 3-cut.

Number of graphs

$\|V\|$	4	6	8	10	12	14	16	18
3-connected	1	1	2	5	14	50	233	1249
4-connected	0	0	1	1	2	4	10	25

Using *plantri* and *Orb* (by Heard), we compute the best possible upper bounds,

Upper bounds

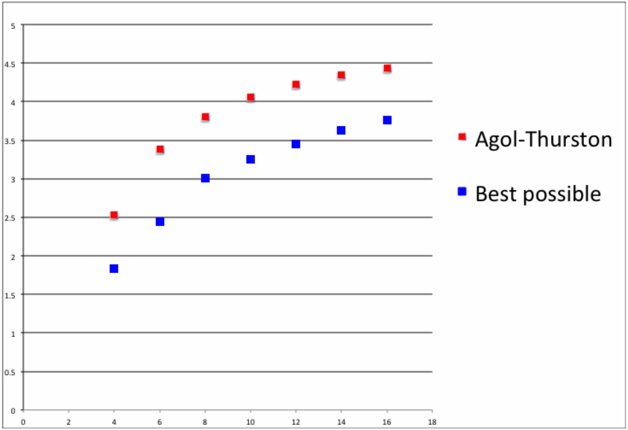

Figure: Upper bounds(U_n)

Further Research

- For given n, which planar graph with n vertices attains the greatest volume?
- Can we compute the formula for the best possible upper bound?

Thank you for your attention.