Free involutions and $\mathbb{Z}_p$-actions on simply-connected 5-manifolds

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The 8th East Asian School of Knots and Related Topics
Jan 2012, Daejeon

Outline

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- free actions of finite groups on the spheres S^n — the spherical space form problem (C. T. C. Wall, Davis, Milgram);
- free $\mathbb{Z}_2$-actions on $\mathbb{C}\mathrm{P}^n$ (PL-category) (Sady).

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Methods - surgery vs modified surgery

The classification of free group actions on a simply-connected manifold M^n is equivalent to the classification of the orbit spaces M/Γ,

$$\begin{array}{ccc} M & \longrightarrow & M \\ \downarrow & & \downarrow \\ M/(\Gamma,\varphi) & \longrightarrow & M/(\Gamma,\psi) \end{array}$$

Therefore the task is to classify certain n-manifolds with $\pi_1 = \Gamma$.

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Methods - surgery vs modified surgery

The theoretical solution to the classification problem is the surgery exact sequence (Browder-Novikov-Sullivan-Wall)

$$\cdots \to L^s_{n+1}(\mathbb{Z}[\pi_1(X)]) \to \mathscr{S}(X) \to [X, G/O] \to L^s_n(\mathbb{Z}[\pi_1(X)]).$$

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In this talk, we will consider free $\mathbb{Z}_2$-actions on some simple 1-connected 5-manifolds.

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Simply-connected 5-manifolds were classified by Smale (the spin case) and Barden (general case) in the 1960's. This is one of the few examples of complete classification of manifolds.

Remark. Smale reduced the problem of classifying spin 5-manifolds to the classification of links

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Simply-connected 5-manifolds with torsion free $H_2(M)$

Theorem (Smale-Barden)

The classification of simply-connected 5-manifolds M^5 with torsion free $H_2(M)$ is given by the following table:

type	$w_2(M) = 0$	$w_2(M) \neq 0$
invariant	$r = \mathrm{rk} H_2(M)$	$r = \mathrm{rk} H_2(M)$
standard form	$\sharp_r(S^2 \times S^3)$	$B\sharp_{(r-1)}(S^2 \times S^3)$

- $B = S^3 \tilde{\times} S^2$ is the twisted S^3-bundle over S^2.
- M^5 is constructed by simple building blocks $S^2 \times S^3$ and B under the connected sum operation. We will see a similar pattern in our classification of 5-manifolds with $\pi_1 = \mathbb{Z}_2$.

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Manifolds with nontrivial fundamental groups

$H_*(\widetilde{M})$ and $\pi_*(M)$ are modules over the group ring $\mathbb{Z}[\pi_1(M)]$. It's natural that the structure of these modules plays an important role in the classification of M.

To have a classification of 5-manifolds with nontrivial fundamental group, the reasonable first step is to look at the simplest nontrivial group $\mathbb{Z}_2$ and the simplest module structures, i. e. torsion free trivial $\mathbb{Z}[\mathbb{Z}_2]$-modules.

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5-manifolds of fibered-type

Definition

An orientable 5-manifold M^5 is said to be of *fibered-type* if $\pi_2(M)$ is torsion free and is a trivial $\mathbb{Z}[\pi_1(M)]$-module.

Motivation and Examples: S^1-bundles over simply-connected 4-manifolds

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We will introduce a classification of 5-manifolds of fibered-type and $\pi_1 = \mathbb{Z}_2$. Namely, we will achieve the following

1. obtain a systems of invariants detecting diffeomorphism types of these manifolds;
2. find all possible relations between these invariants;
3. write down all such manifolds explicitly.

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Types

- similar to the $\pi_1 = 0$ case, manifolds with $\pi_1(M) = \mathbb{Z}_2$ can be divided into three w_2-types:

type I	type II	type III
$w_2(\widetilde{M}) \neq 0$	$w_2(M) = 0$	$w_2(M) \neq 0$ $w_2(\widetilde{M}) = 0$

- by universal coefficient theorem

$$0 \to \mathrm{Ext}(H_1(M), \mathbb{Z}/2) \to H^2(M; \mathbb{Z}/2) \to \mathrm{Hom}(H_2(M), \mathbb{Z}/2) \to 0$$

we see that

- type III $\Leftrightarrow$ $w_2(M) \neq 0$ and $w_2(M) \in \mathrm{Ext}(H_1(M), \mathbb{Z}/2)$;
- type II $\Leftrightarrow$ $w_2(M) = 0$;
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System of invariants

Theorem (Hambleton-S.)

Two smooth, closed, orientable fibered type 5-manifolds M and M' with $\pi_1 = \mathbb{Z}_2$ and torsion free H_2 are diffeomorphic if and only if

1. *they have the same w_2-type,*
2. rank $H_2(M) =$ rank $H_2(M')$,
3. $[P] = [P'] \in \Omega_4^{\mathrm{Pin}^\dagger}/\pm$, *where P (reps. P') is characteristic submanifold of M (reps. M') and $\dagger = c, -, +$ for w_2-types* I, II, III *respectively.*

Characteristic submanifolds

If we decompose $\widetilde{M} = A \cup TA$ such that $\partial A = \partial TA = \widetilde{P}$, then $P = \widetilde{P}/T$ is called the *characteristic submanifold* of M^5.

The universal example is $M = \mathbb{RP}^n$ and $P = \mathbb{RP}^{n-1}$.

In general, let $f : M \to \mathbb{RP}^n$ be the classifying map of the universal cover such that $f \pitchfork \mathbb{RP}^{n-1}$, then $P = f^{-1}(\mathbb{RP}^{n-1})$.

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In general, let $f : M \to \mathbb{R}\mathrm{P}^n$ be the classifying map of the universal cover such that $f \pitchfork \mathbb{R}\mathrm{P}^{n-1}$, then $P = f^{-1}(\mathbb{R}\mathrm{P}^{n-1})$.

Pin†-structures

Central extension of $O(n)$:

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}^+(n) \to O(n) \to 1.$$

A real vector bundle ξ of rank n over X admits a Pin$^+(n)$-structure $\Leftrightarrow w_2(\xi) = 0 \Leftrightarrow \xi \oplus 3\det\xi$ admits a spin structure.

$\{\text{Pin}^+\text{-structures on } \xi\} \overset{1:1}{\longleftrightarrow} H^1(X;\mathbb{Z}_2)$.

n-dimensional smooth manifolds with Pin$^+$-structures on the tangent bundle form a bordism group $\Omega_n^{\mathrm{Pin}^+}$.

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Values of the cobordism groups

The $\mathrm{Pin}^\dagger$-cobordism groups have values and generators as follows (Kirby-Taylor)(Hambleton-Su):

$\dagger$	$\Omega_4^{\mathrm{Pin}^\dagger}$	generators
c	$\mathbb{Z}_8 \oplus \mathbb{Z}_2$	$\mathbb{R}\mathrm{P}^4$, $\mathbb{C}\mathrm{P}^2$
$+$	$\mathbb{Z}_{16}$	$\mathbb{R}\mathrm{P}^4$
$-$	0	–

Relations between the invariants

The invariants are subject to certain relations. Denote $r = \operatorname{rank} H_2(M)$, $q = [P] \in \Omega_4^{\mathrm{Pin}^+}/\pm = \{0, 1, \ldots, 8\}$ and $(q, s) = [P] \in \Omega_4^{\mathrm{Pin}^c}/\pm = \{0, 1, \ldots, 4\} \times \{0, 1\}$. Then we have

Theorem

type	*relation*
I	$q + s + r \equiv 1 \pmod 2$
II	$r \equiv 1 \pmod 2$
III	$q + r \equiv 1 \pmod 2$

Standard forms

Theorem

Every closed smooth orientable fibered type 5-manifold M with $\pi_1 = \mathbb{Z}_2$ and $H_2 = \mathbb{Z}^r$ is diffeomorphic to exactly one of the following standard forms:

$$\textit{type I}: X^5(q)\,\sharp_{S^1}(S^2 \times \mathbb{R}\mathrm{P}^3)\,\sharp_{S^1}((\sharp_k\, S^2 \times S^2) \times S^1), \quad r = 2k + (5 + (-1)^q)/2,\ q \in \{0, \ldots, 4\};$$

$$X^5(q)\,\sharp_{S^1}(\mathbb{C}\mathrm{P}^2 \times S^1)\,\sharp_{S^1}((\sharp_k\, S^2 \times S^2) \times S^1), \quad r = 2k + (3 + (-1)^q)/2,\ q \in \{0, \ldots, 4\};$$

$$\textit{type II}: (S^2 \times \mathbb{R}\mathrm{P}^3)\,\sharp_{S^1}((\sharp_k\, S^2 \times S^2) \times S^1),\ r = 2k + 1;$$

$$\textit{type III}: X^5(q)\,\sharp_{S^1}((\sharp_k\, S^2 \times S^2) \times S^1) \quad r = 2k + (1 + (-1)^q)/2,\ q \in \{0, \ldots, 8\}.$$

Construction — connected-sum along S^1

Let $E = S^1 \times D^4$. Choose embeddings of E into M_i representing the nontrivial element in π_1. Then we define

$$M_1 \sharp_{S^1} M_2 := (M_1 - E) \cup_\partial (M_2 - E).$$

Note that since $\pi_1 SO(4) \cong \mathbb{Z}_2$, there are actually two possibilities to form $\sharp_{S^1}$. One needs to fix additional structures on M_1 and M_2 to make the choice unique.

The characteristic submanifold of $M_1 \sharp_{S^1} M_2$ is $P_1 \sharp_{S^1} P_2$, which corresponds to the addition in the bordism group $\Omega_4^{\mathrm{Pin}^\dagger}$.

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Building blocks — fake $\mathbb{R}P^5$

In the smooth or PL category, there exist 4 fake $\mathbb{R}P^5$'s: $X^5(q) \simeq \mathbb{R}P^5$ $(q = 1, 3, 5, 7)$.

Explicit construction (Geiges-Thomas): consider

$$V : z_0^q + z_1^2 + z_2^2 + z_3^2 = 0 \text{ in } \mathbb{C}^4$$

$$\Sigma_q^5 := V \cap S^7 \cong \begin{cases} S^5 & q \text{ odd} \\ S^2 \times S^3 & q \text{ odd} \end{cases} \quad \text{(Brieskorn varieties)}$$

Involution

$$T : \mathbb{C}^4 \to \mathbb{C}^4 \ (z_0, z_1, z_2, z_3) \mapsto (z_0, -z_1, -z_2, -z_3).$$

$T|\Sigma_q^5$ is a free involution, $X^5(q) = \Sigma_q^5 / T$ is a fake $\mathbb{R}P^5$ when q is odd. $X^5(1) = \mathbb{R}P^5$.

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Point of view of free involutions

For the manifolds M under consideration, $H_2(\widetilde{M})$ is torsion free. Therefore, by the classification of 1-connected 5-manifolds, $\widetilde{M} \cong \sharp_r(S^2 \times S^3)$ or $\widetilde{M} \cong B\sharp_{r-1}(S^2 \times S^3)$.

From this point of view, the above theorems give a classification of orientation preserving free involutions on $\sharp_r(S^2 \times S^3)$ and $B\sharp_{r-1}(S^2 \times S^3)$, where the action on H_2 is trivial.

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Application — contact structures

Classification results can also be useful in studying the existence problem for geometric structures on fibered type 5-manifolds.

- Geiges and Thomas showed that a closed, orientable 5-manifold with $\pi_1 = \mathbb{Z}_2$, such that w_2 vanishes on homology, admits contact structures;
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Let M^5 be an S^1-bundle over a 1-connected 4-manifold X^4 with Chern class $c_1 = m\cdot$primitive. Then M^5 is of fibered-type with $\pi_1(M) = \mathbb{Z}_m$.

- $m = 1$, M is 1-connected. Haibai Duan and Chao Liang gave a classification of M, making use of the classification of 1-connected 5-manifolds.
- $m = 2$, $\pi_1(M) = \mathbb{Z}_2$. An explicit classification of M can be given by making use of the above classification theorems .

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Topological case

The method also applies to topological manifolds and topological actions. The structure is similar, but some new phenomenon appears:

We constructed a topological manifold $*(S^2 \times \mathbb{RP}^3)$, which is homotopy equivalent to $S^2 \times \mathbb{RP}^3$ but non-smoothable.

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Free involutions on $S^2 \times S^3$

Now we look at an example of free involutions on a 5-manifold M^5, where all possible $\mathbb{Z}[\mathbb{Z}_2]$-module structures of $\pi_2(M)$ are taken into account .

We will give a classification of all smooth involutions on $S^2 \times S^3$.

Examples I— obvious ones

There are obvious "linear" actions

- $$T_i \colon S^2 \times S^3 \to S^2 \times S^3, \quad (x, y) \mapsto (-x, \tau_i(y))$$
  where τ_i is the reflection with i (-1)-eigenvalues $i = 0, 1, 2, 3$. Denote $Y_i = S^2 \times S^3 / T_i$.
- $$T'_j \colon S^2 \times S^3 \to S^2 \times S^3, \quad (x, y) \mapsto (\tau'_j(x), -y)$$
  where τ'_j is the reflection with j (-1)-eigenvalues $j = 0, 1, 2$. Denote $Z_j = S^2 \times S^3 / T'_j$.

Examples II— actions on Brieskorn varieties

Recall:

$$V : z_0^q + z_1^2 + z_2^2 + z_3^2 = 0 \ \text{ in } \ \mathbb{C}^4$$

$$\Sigma_q^5 := V \cap S^7 \cong \begin{cases} S^5 & q \text{ odd} \\ S^2 \times S^3 & q \text{ odd} \end{cases} \quad \text{(Brieskorn varieties)}$$

Involution

$$T : \mathbb{C}^4 \to \mathbb{C}^4 \ \ (z_0, z_1, z_2, z_3) \mapsto (z_0, -z_1, -z_2, -z_3).$$

$T|\Sigma_q^5$ is a free involution, denote the quotient space by Σ_q^5/T, $q = 2, 4, 6, 8$.

Let T be a smooth free involution on $S^2 \times S^3$, $M^5 = S^2 \times S^3/T$ be the quotient space. Then as a $\mathbb{Z}[\mathbb{Z}_2]$-module, $\pi_2(M) = \mathbb{Z}_+$ or $\mathbb{Z}_-$.

- if $\pi_2(M) = \mathbb{Z}_+$, then this falls into the situation we have dealt with;
- if $\pi_2(M) = \mathbb{Z}_-$, denote the second stage of the Postnikov tower of M by $P_2(M)$. Then $P_2(M)$ is a fibration over $K(\mathbb{Z}_2, 1)$ with fiber $K(\mathbb{Z}, 2)$;
- the k-invariant lies in $H^3(\mathbb{Z}_2; \mathbb{Z}_-) = \mathbb{Z}_2$, there are exactly two such fibrations, denoted by P and Q.

With these notations, the classification results are given as follows:

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With these notations, the classification results are given as follows:

Theorem

Let M^5 be a smooth 5-manifold with $\pi_1(M) \cong \mathbb{Z}_2$ and universal cover $\widetilde{M} \cong S^2 \times S^3$. The classification of M up to diffeomorphism is given in the following tables

- *If $\pi_2(M) \cong \mathbb{Z}_+$, then M is orientable.*

$w_2(M) = 0$	$w_2(M) \neq 0$
$S^2 \times \mathbb{RP}^3$	$X^5(q), \quad q = 0, 2, 4, 6, 8$

- *If $\pi_2(M) \cong \mathbb{Z}_-$ and M is orientable, then $P_2(M) = Q$.*

$w_2(M) = 0$	$w_2(M) \neq 0$
$Y_1, \; Y_1'$	$\Sigma^5_q / T, \quad q = 0, 2, 4, 6, 8$

Theorem (continued)

- *If $\pi_2(M) \cong \mathbb{Z}_-$ and M is nonorientable.*

	$w_2(M) = 0$	$w_2(M) \neq 0$
$P_2(M) = P$	Z_1	$-$
$P_2(M) = Q$	Y_2	$S^3 \times \mathbb{RP}^2$

All the manifolds in the classification tables (Y_i, Z_j, Σ^5_q/T) come from explicit involutions on $S^2 \times S^3$, except for Y_1', which I constructed by surgery and I don't know the corresponding involution on $S^2 \times S^3$.

Y_1 and Y_1' are distinguished by a KO-characteristic number

$$\langle f^* u, [Y]_{ko} \rangle \in ko_1 = \mathbb{Z}_2.$$

Y_1 is not homotopy equivalent to Y_1'.

Free $\mathbb{Z}_p$-actions

Let T be a free $\mathbb{Z}_p$-action on $X = \sharp_k S^2 \times S^3$, then $H_2(X) = \mathbb{Z}^k$ is a torsion free $\mathbb{Z}[\mathbb{Z}_p]$-module.

The first step toward a classification of $\mathbb{Z}_p$-actions is the knowledge of the classification of torsion free $\mathbb{Z}[\mathbb{Z}_p]$-modules.

Let $\Lambda = \mathbb{Z}[\mathbb{Z}_p]$, $D = \mathbb{Z}[e^{2\pi i/p}]$ (cyclotomic ring), then D is an ideal of Λ

$$0 \to D \to \Lambda \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Also

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torsion free $\mathbb{Z}[\mathbb{Z}_p]$-modules

Theorem (Reiner, 1957)

Let M be an indecomposable torsion free Λ-module, then

$$M \cong \begin{cases} \mathbb{Z} & \text{the trivial module} \\ I & I \text{ is an ideal of } D \\ I \oplus \mathbb{Z} & I \text{ an ideal of } D \text{ (direct sum of abelian groups)} \end{cases}$$

The number of indecomposable $\mathbb{Z}[\mathbb{Z}_p]$-modules $= 2h(p) + 1$, where

$$h(p) = |\{\text{ideals in } D\}/\cong|$$

is called the class number of the prime number p.

- $h(p) = 0$ for $p \leq 19$, $h(23) = 3, \cdots$
- $411, 322, 824, 001 | h(97)$
- in general hard to compute.

For comparison, recall:

- the number of irreducible $\mathbb{C}$-representations $= p$
- the number of irreducible $\mathbb{R}$-representations $= (p+1)/2$
- the number of irreducible $\mathbb{Q}$-representations $= 2$

Question

Which finitely generated torsion free $\mathbb{Z}[\mathbb{Z}_p]$-module can be realized by a free $\mathbb{Z}_p$-action on $\sharp_k S^2 \times S^3$?

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Conjecture

Every finitely generated torsion free $\mathbb{Z}[\mathbb{Z}_p]$-module can be realized by a free $\mathbb{Z}_p$-action on $\sharp_k S^2 \times S^3$.

Remark. Every finitely presented group can be realized as the fundamental group of a closed manifold of dimension ≥ 4.

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Possible approach:

1. Given a finitely generated torsion free $\mathbb{Z}[\mathbb{Z}_p]$-module M, we need to construct a finite (simple) Poincaré-complex X^5, such that $\pi_1(X) = \mathbb{Z}_p$, $\pi_2(X) \cong M$, and $w_2(X) = 0$.
2. If the Spivak normal fibration has a vector bundle reduction, then there is a 5-manifold Y homotopy equivalent to X. This is seen by

$$\cdots \to \mathscr{S}(X) \to [X, G/O] \xrightarrow{\theta} L_5^s(\mathbb{Z}[\mathbb{Z}_p])$$

   and $L_5^s(\mathbb{Z}[\mathbb{Z}_p]) = 0$.
3. By the classification of 1-connected 5-manifolds, the universal cover $\widetilde{Y} \cong \sharp_k S^2 \times S^3$ and the deck transformation is a free $\mathbb{Z}_p$-action.

THANKS!