Exceptional surgeries on components of two-bridge links

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10:15

1. Introduction

Classification of 3-manifolds

As a consequence of

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Every closed orientable 3-manifold is;

- Reducible (containing essential S^2)
- Toroidal (containing essential torus)
- Seifert fibered (foliated by circles)
- Hyperbolic ($^{\exists}$ Riem.metric of curv.-1)

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- Relate Geometric & Topological invariants
  (e.g., Volume conjecture . . .)
- Study the Relationships between 3-mfds.
  (e.g., **Dehn surgery** . . .)
  (⇑ **Today!**)

Dehn surgery on link

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(i.e., $M-$(open tubular nbd of L))

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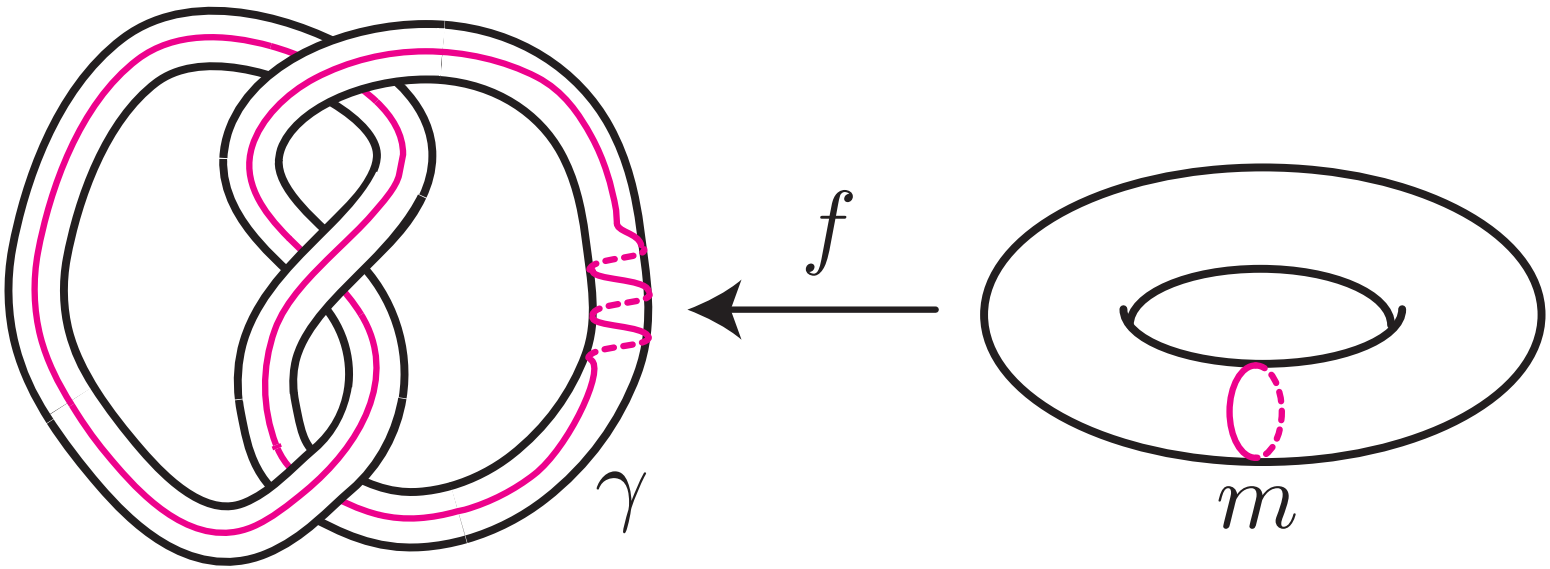
Gluing solid torus V to $E(L)$ along slope γ



Dehn surgery vs Classification

Experimentally we can see that

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Observation

The types of $E(L)$ and of the surgered mfd. agree generically.

Motivation for studying Dehn surgery

a hyperbolic link $:= M - L$ is hyperbolic

Hyperbolic Dehn Surgery Theorem [Thurston]

On each component of a hyperbolic link, **all** but only finitely many surgeries give hyperbolic 3-manifolds.

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Exceptional surgery:=

Dehn surgeries on a hyperbolic link giving non-hyperbolic mfds

Exceptional surgery

Ultimate Goal

Completely classify the exceptional surgeries on hyperbolic links in the 3-sphere S^3.

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Problem

Completely classify the exceptional surgeries on hyperbolic 2-bridge links in the 3-sphere S^3.

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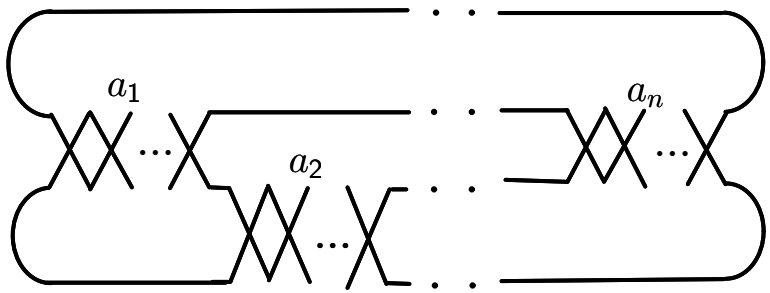
Completely classify the exceptional surgeries on hyperbolic 2-bridge links in the 3-sphere S^3.

Today's Goal

Completely classify the exceptional surgeries on a component of hyperbolic 2-bridge links in S^3.

2-bridge link

A 2-component link admitting a diagram with two maxima and minima.



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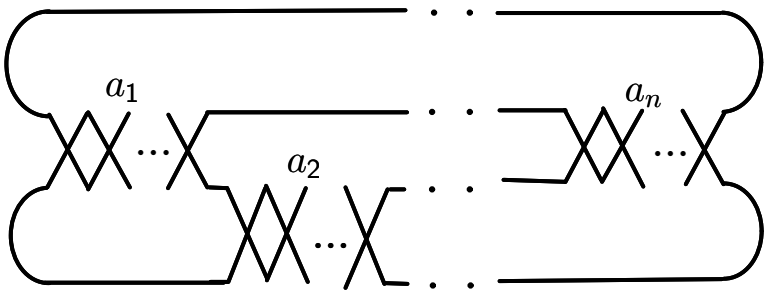

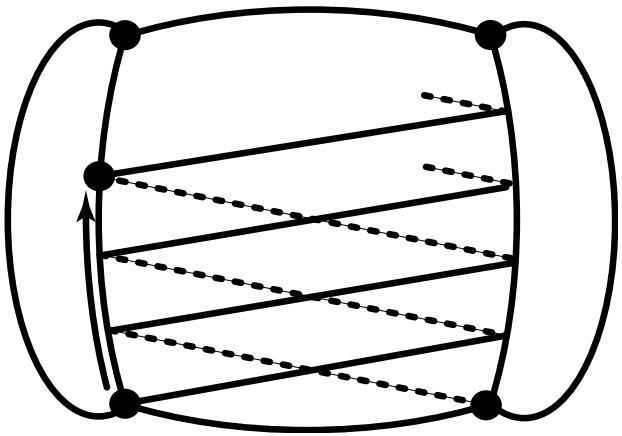

2-bridge links are parametrized by rational numbers $\mathbb{Q}$

We denote by $L_{p/q}$ the link with p/q.

10:25

2. Known Facts

Known facts (2-bridge knots)

Brittenham-Wu (2001)

Exceptional surgeries on 2-bridge knots are completely classified.

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Brittenham-Wu (2001)

Exceptional surgeries on 2-bridge knots are completely classified.

For example, they showed that only knots $K_{[b_1,b_2]}$ have exceptional surgeries.

Notation:

$p/q = [a_1, a_2, \cdots, a_n]$, continued fraction

Known facts (2-bridge links)

Let M_L be a 3-mfd obtained by a Dehn surgery on a component of a 2-bridge link L.

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Theorem [Wu (1999)]

If M_L contains an essential disk, annulus, or 2-sphere, then L is equivalent to $L_{[b_1,b_2]}$.

Known facts (2-bridge links)

Note that M_L can be regarded as a knot complement in some lens space.

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Theorem [Goda-Hayashi-Song (2009)]

The following are obtained:

A complete classification of L for which M_L is a non-trivial, non-core torus knot exterior or a cable knot exterior.

A necessary condition of L for which M_L is a prime satellite knot exterior.

10:30

3. Result

Notation (surgery slope)

Let L be a 2-bridge link.

$L(r)$ denotes the manifold obtained by Dehn surgery on a compo. of L along slope r;

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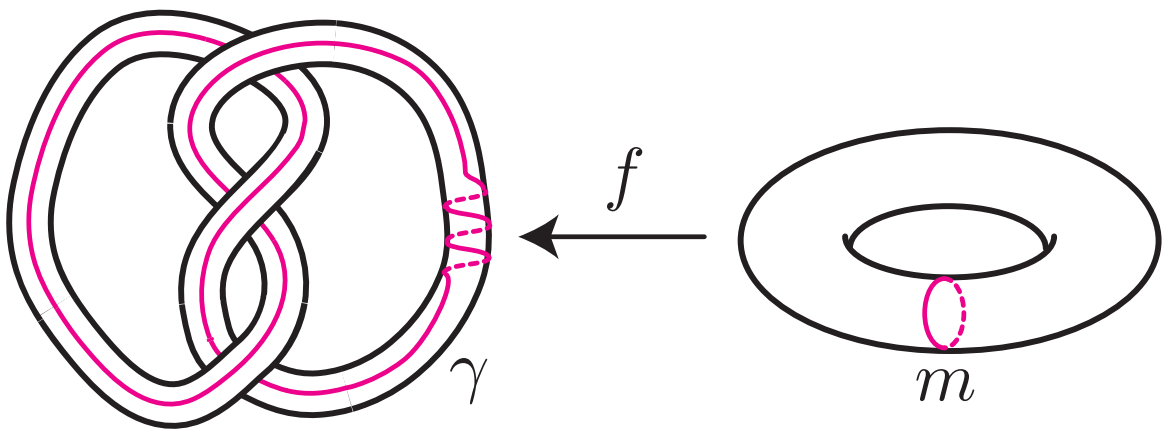
$L(r)$ denotes the manifold obtained by Dehn surgery on a compo. of L along slope r;

i.e., the slope given by [f(meridian of V)] corresponds to $r \in \mathbb{Q}$.



Classification of exceptional surgery

Note: $L(r)$ is a 3-mfd. with torus boundary.

Classification

If $L(r)$ is non-hyperbolic, then it contains an essential

- disk D
- annulus A
- sphere S
- torus T

Theorem (toroidal)

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$L(r)$ contains neither essential D nor S.

$L(r)$ contains an essential torus if and only if

$$\boxed{L \cong L_{[2w,v,2u]} \quad \& \quad r = -w - u}$$

with
1. $w = 1, u = -1, |v| \geq 2$
2. $w \geq 2, |u| \geq 2, |v| = 1$
3. $w \geq 2, |u| \geq 2, |v| \geq 2$

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3. $w \geq 2, |u| \geq 2, |v| \geq 2$

In all the cases, $L(r)$ is never SFS, and
$L(r)$ is a graph mfd if and only if
u, v, w satisfy 1st & 2nd conditions.

Theorem (continued)

If $L(r)$ contains an essential annulus,
but contains no essential tori,
then $L(r)$ is a Seifert fibered space.

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then $L(r)$ is a Seifert fibered space.
$L(r)$ is a Seifert fibered space if and only if

1. $L \cong L_{[3,2u+1]}$ & $r = u$
2. $L \cong L_{[2w+1,3]}$ & $r = -w - 1$
3. $L \cong L_{[3,-3]}$ & $r = -1$
4. $L \cong L_{[2w+1,2u+1]}$ & $r = -w + u$

for $w \geq 1, u \neq 0, -1$.

10:35

4. Outline of Proof

Essential surfaces

$L_{p/q}(r)$ is non-hyperbolic

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$\Downarrow$

$E(L_{p/q})$ contains

an essential properly embedded surface F

with non-empty boundary of genus $g \leq 1$

Cases

Case 1: F is meridionally incompressible

Case 1A: $\partial F \cap \partial N(K_2) = \emptyset$

Case 1Aa: $|v| = 1$

Case 1Ab: $|v| \neq 1$

Case 1B: $\partial F \cap \partial N(K_2) \neq \emptyset$

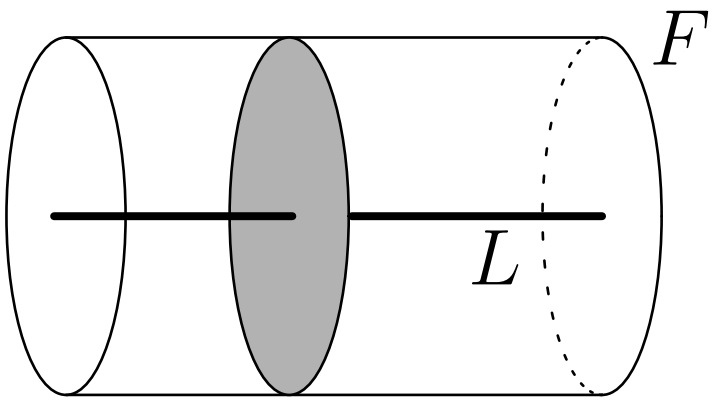
Case 1Ba: $r_2 \neq 1/0$

Case 1Bb: $r_2 = 1/0$

Case 2: F is meridionally compressible

meridionally incompressible case

Case 1: F is meridionally incompressible



No such meridionally compressing disk

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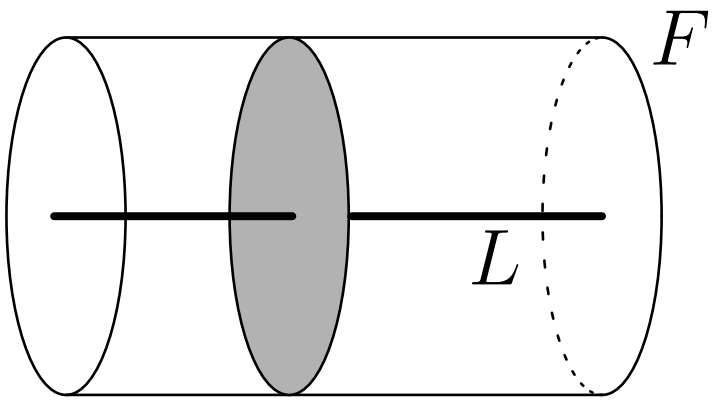

No such meridionally compressing disk

Meridionally incompressible essential surfaces in 2-bridge link exteriors are classified by [Floyd-Hatcher, 1988].

Case 1A

Set $L = K_1 \cup K_2$, and
$L(r)$ is obtained by r-surgery on K_1.
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Case 1A: $\partial F \cap \partial N(K_2) = \emptyset$

Then, by [FH + GHS(Lem12.1)], we have

$$g = 1, \quad L \cong L_{[2w,v,2u]}, \quad r = -w - u$$

for $w \geq 1, u, v \neq 0$.

Case 1A (continued)

Case 1Aa: $|v| = 1$

The following are shown in [GHS, Section 11].

Case 1A (continued)

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The following are shown in [GHS, Section 11].

Only when $L \cong L_{[2w,\pm 1,2u]}$
with $w \geq 2$ & $u \neq 0, -1, -2$,

$L(-w-u)$ contains an essential torus,
and actually is a graph manifold.

Case 1A (continued)

Case 1Ab: $|v| \neq 1$

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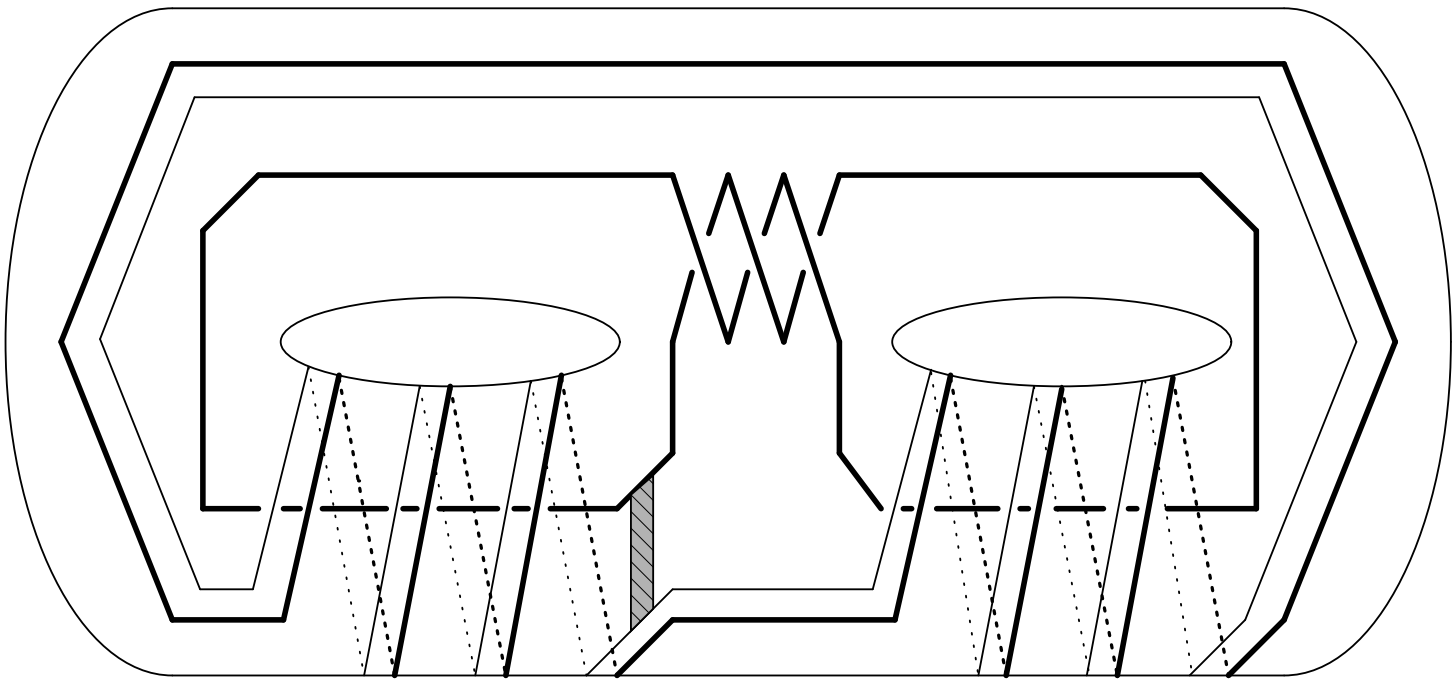
Case 1Ab: $|v| \neq 1$

$L_{[2w,v,2u]}(-w-u)$ contains an essential torus, and is not a graph mfd. (c.f. [Wu] & [GHS])



Case 1Ba

Case 1B: $\partial F \cap \partial N(K_2) \neq \emptyset$

$$\partial F \cap \partial N(K_2) \neq \emptyset$$
$$\Rightarrow L(r) \not\supset S, T \Rightarrow L(r) \supset D, A$$

Case 1Ba

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$L(r) \not\supset D$ (by [Miyazaki-Motegi])

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r_2: the slope of ∂F on $\partial N(K_2)$.

Case 1Ba: $r_2 \neq 1/0$

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r_2: the slope of ∂F on $\partial N(K_2)$.

Case 1Ba: $r_2 \neq 1/0$

$L(r)$ is a SFS (i.e., $L(r) \supset A$, $\not\supset T$)

if and only if the conditions are satisfied described in Theorem (SFS) by [Wu+GHS(Thm11.1)].

Case 1Bb

Case 1Bb: $r_2 = 1/0$

$$L(r) \supset D, A$$
$$\Rightarrow g = 0$$
$$\& \ |\partial F \cap \partial N(K_2)| \leq 2$$

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Case 1Bb: $r_2 = 1/0$

$L(r) \supset D, A$
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Lemma

$E(L)$ contains an meri. incomp. ess. surface F with $g = 0$, $|\partial F \cap \partial N(K_2)| \leq 2$, $r_2 = 1/0$

iff $\boxed{L \cong L_{[2,n,-2]} \text{ \& } \partial\text{-slope 0 on } \partial N(K_1)}$

with $|n| \geq 2$, & F: an ess. 2 punctured disk.

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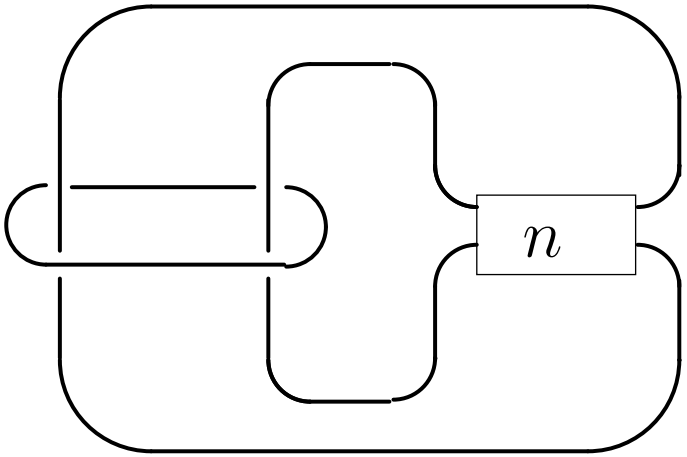

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$\Downarrow$ (by Lemma)

$L \cong L_{[2,n,-2]}$ & ∂-slope 0 on $\partial N(K_1)$

with $|n| \geq 2$, & F: an ess. 2 punctured disk.

$\square$ (Theorem)

5. Next Problem

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We have a potential approach to attack;
via 3 steps as follows:

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Floyd-Hatcher machinery can be used:

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but is much complicated, and
the task seems technically very hard...

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2) Give restrictions for Seifert surgeries

Y.-Q. Wu,
Dehn surgery on arborescent links,
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Wu constructed persistent laminations
in 2-bridge link exterior of length ≥ 3.

They or their refinements could show
absence of Seifert surgeries on
2-bridge link exterior of length ≥ 3.

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could give such a classification,

since exceptional surgeries
on 2-bridge links of length two
is induced from those on the 4-chain link.

Acknowledgments

Thank you !!