Invariants of Links from the Homotopy Groups of the 3-Sphere

Fengchun Lei (雷逢春)
(Joint with Fuquan Fang and Jie Wu)

School of Mathematical Sciences
Dalian University of Technology, Dalian，CHINA

fclei@dlut.edu.cn

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Contents

Basic definitions

Let M be a compact 3-manifold. Let C be a circle. By an n-link L will be meant an ordered collection $(l_1, \cdots, l_n)$ of maps $l_i : C \to M$, where the images $l_1(C), \cdots, l_n(C)$ are to be disjoint. A link will be called proper if the maps $l_1, \cdots, l_n$ are all embeddings.

Two links L and L' will be called homotopic if there exist homotopies h_{it}, between the maps l_i and the maps l'_i so that the sets $h_{1t}(C), \cdots, h_{nt}(C)$ are disjoint for each value of t, that is, L can be transformed to L' by a series of self-crossing changes and ambient isotopies.

Note that in a link homotopy, we allow crossing changes in L as long as both strands of the crossing belong to the same component of L (self-crossing change).

Homotopically trivial links

A link L is homotopically trivial if it is homotopic to some $(l'_1, \cdots, l'_n)$, where the components $l'_1(C), \cdots, l'_n(C)$ consist of single points. A proper link L_n is trivial if each component l_i of L_n bounds a disk D_i in the manifold and $D_i \cap D_j = \emptyset$ for all $i \neq j$. A one component link is called a knot. Clearly, a knot is always homotopically trivial.

A n-link $L = \{l_1, \cdots, l_n\}$ is almost homotopically trivial (or Brunnian) if $L \setminus \{l_i\}$ is homotopically trivial for any $1 \leq i \leq n$.

A usual n-link $L = \{l_1, \cdots, l_n\}$ is Brunnian if $L \setminus \{l_i\}$ is a trivial link for any $1 \leq i \leq n$.

Brunnian type of links

Let $L = \{l_1, \cdots, l_n\}$ be an n-link in M, $M_L = M \smallsetminus L$, the link complement. Let K be a knot in M_L.

Definition

K is called a Brunnian type of knot with respect to L if K is homotopically trivial in $M_{L \smallsetminus \{l_i\}}$ for any $1 \leq i \leq n$.

Namely, K is a Brunnian type knot with respect to L if and only if by removing any i th component of L, the knot K can be moved in $M_{d_i L}$ by self-crossing changes and ambient isotopies so that it separates away from $d_i L$ and is homotopic to the homotopically trivial knot.

An Example of a Brunnian type knot

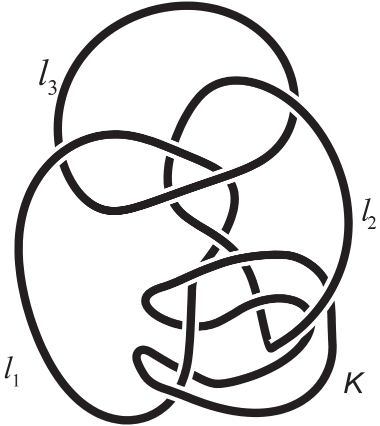

Figure: $L = \{l_1, l_2, l_3\}$ is a Hopf link of 3 components in S^3, and K is a Brunnian type of knot with respect to L.

The normal closure of the meridian

Let $L = \{l_1, l_2, \ldots, l_n\}$ be an n-link in a closed orientable irreducible 3-manifold M, where l_i is the i th component of L. Set $G(L) = \pi_1(M \setminus L)$.

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Let $d_i L = \{l_1, l_2, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n\}$ be the $(n-1)$-link obtained by removing the ith link component of L.

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Let $d_i L = \{l_1, l_2, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n\}$ be the $(n-1)$-link obtained by removing the ith link component of L.

The inclusion of link complement $M \setminus L \hookrightarrow M \setminus d_i L$ induces a group homomorphism $d_i \colon G(L) \longrightarrow G(d_i L)$. Let $A_i = A(L, l_i) = \mathrm{Ker} d_i$.

Note that A_i is the normal closure of the meridian of l_i in $G(L)$.

An equivalent description of Brunnian type links

Let

$$A(L, L) = \bigcap_{i=1}^{n} A(L, l_i), \tag{1}$$

i.e. the intersecting subgroup of all $A(L, l_i)$, $1 \leq i \leq n$.

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i.e. the intersecting subgroup of all $A(L, l_i)$, $1 \leq i \leq n$.

Let K be a knot in M_L, the complement of L in M. It is not hard to check that

Proposition

K is a Brunnian type of knot with respect to L if and only if $[K] \in \bigcap_{i=1}^{n} A(L, l_i)$.

Milnor's link invariants

The study of the fundamental group of the complement of a link has led to link invariants in many ways. The Milnor $\overline{\mu}$-invariants are an important class of such invariants.

Milnor introduced the notion of link-homotopy in 1954. Some link-homotopy invariants of links were introduced there which turned out to be a subcollection of the $\overline{\mu}$-invariants in 1957, which have been regarded as the higher order linking numbers. The methodology in Milnor's work can be briefly reviewed as follows:

As before, let $L = \{l_1, l_2, \ldots, l_n\}$ be an n-link in a closed orientable irreducible 3-manifold M, and $G(L) = \pi_1(M \smallsetminus L)$. Let $d_i L = \{l_1, l_2, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n\}$ be the $(n-1)$-link obtained by removing the i th link component of L.

Milnor's link invariants

The inclusion of link complement $M \setminus L \hookrightarrow M \setminus d_i L$ induces a group homomorphism $d_i \colon G(L) \longrightarrow G(d_i L)$. Let $A_i = A(L, l_i) = Ker d_i$, $1 \leq i \leq n$.

Then the Milnor's homotopy link group $\mathcal{G}(L)$ is defined by

$$\mathcal{G}(L) = G(L) / \prod_{i=1}^{n} [A_i, A_i],$$

where $[A_i, A_i]$ is the commutator subgroup, $1 \leq i \leq n$.

Milnor showed that $\mathcal{G}(L)$ is invariant under a link homotopy. Moreover, Milnor described a representation of elements in $\mathcal{G}(L)$ for L in S^3.

Milnor's link invariants

For an n-component link $L = \{l_1, \cdots, l_n\}$ in S^3, the Milnor numbers, denoted by $\mu_L(\Gamma)$, are specified by a multi-index Γ, where the entries of Γ are chosen from $\{1, \cdots, n\}$. We may compute Milnor's numbers in the following way.

Let $G = \pi_1(M_L)$, and let G_q be the qth subgroup of the lower central series of G. Here $G_1 = G$, and inductively, G_q is generated by all commutators $aba^{-1}b^{-1}$ with $a \in G$ and $b \in G_{q-1}$.

There is a presentation of G/G_q with n generators, given by the meridians m_i, $1 \leq i \leq n$, of the components of L. So for $1 \leq i \leq n$, a longitude l_i of the ith component of L is expressed modulo G_q as a word in the m_i's.

Milnor's link invariants

The Magnus expansion $E(l_j)$ of l_j is the formal power series in noncommuting variables $X_1, \cdots, X_n$ obtained by substituting m_i by $1 + Xi$ and m_i^{-1} by $1 - X_i + X_i^2 - X_i^3 + \cdots$ for $1 \leq i \leq n$.

Let $G = \pi_1(E(L))$, and let G_q be the qth subgroup of the lower central series of G. Here $G_1 = G$, and inductively, G_q is generated by all commutators $aba^{-1}b^{-1}$ with $a \in G$ and $b \in G_{q-1}$.

We have a presentation of G/G_q with n generators, given by the meridians m_i, $1 \leq i \leq n$, of the components of L. So for $1 \leq i \leq n$, a longitude l_i of the ith component of L is expressed modulo G_q as a word in the m_i's. Milnor's numbers are the coefficients of the monomials in the Magnus expansions of the longitudes l_j.

Milnor's link invariants

Specifically, given a multi-index Γ with entries from $\{1, \cdots, n\}$, the number $\mu_L(i_1 \cdots i_r j)$ is the coefficient of $X_{i_1} \cdots X_{i_r}$ in the Magnus expansion of l_j. That is,

$$E(l_j) = 1 + \sum \mu_L(i_1 \cdots i_r j) X_{i_1} \cdots X_{i_r}.$$

The Milnor's $\overline{\mu}_L(\Gamma)$ number is defined to be $\mu_L(\Gamma)$ modulo the greatest common divisor of all lower order Milnor numbers. Milnor showed that if no index is repeated in the multi-index Γ, then $\overline{\mu}_L(\Gamma)$ is an invariant of link homotopy. $\overline{\mu}_L = \{\overline{\mu}_L(\Gamma)\}$ are the Milnor's $\overline{\mu}$-invariants, which is 1-1 corresponding to an element in $\mathcal{G}(L)$.

Milnor's link invariants

There have been fruitful study on Milnor's μ-invariants (as well as $\bar{\mu}$-invariants) for links.

V. G. Turaev and R. Porter showed that the Milnor's $\overline{\mu}$-invariants coincide with cohomological invariants of the link complement.

The classification of links up to link-homotopy was achieved by Milnor for links with two and three components, it was only after more than thirty years that Levine accomplished such a classification for links with four components. The classification of links up to link-homotopy was finally completed by Habegger-Lin in 1990 (via string links), and they give an algorithm which can decide whether two elements in the homotopy link group are equivalent.

Some simple examples

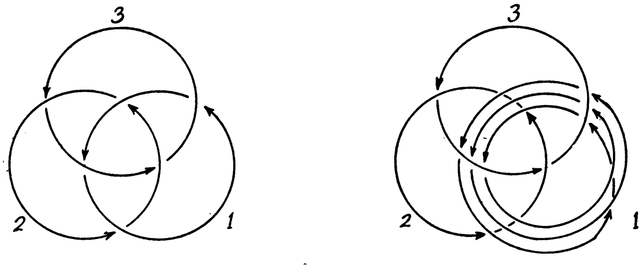

Figure: $\mu(1,23) = 1$ and $\mu(1,23) = 3$

Some Examples of Brunnian links

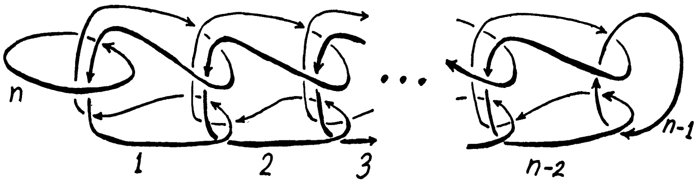

Figure: $\mu(12\cdots n-2, n-1n) = 1$ and $\mu(i_1 i_2 \cdots i_{n-2}, n-1n) = 0$ for all other permutations of $1, 2, \cdots, n-2$

Definition of Commutator quotient group

For a n-link $L = \{l_1, l_2, \ldots, l_n\}$ in a closed orientable 3-manifold M, we have seen that the homotopy link group

$$\mathcal{G}(L) = G(L) / \prod_{i=1}^{n} [A_i, A_i]$$

is invariant under link homotopy. In the following, we will derive another link homotopy invariant from $G(L)$.

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As before, let $L = \{l_1, l_2, \ldots, l_n\} \subset M$,
$d_i L = \{l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n\}$,
and $A_i = A(L, l_i) = Ker d_i$, for $1 \leq i \leq n$.

Definition of Commutator quotient group

Let $L' = \{l_{i_1}, l_{i_2}, \ldots, l_{i_t}\}$ be a sublink of L. Let

$$A(L, L') = \bigcap_{j=1}^{t} A(L, l_{i_j}), \tag{2}$$

i.e. the intersecting subgroup of $A(L, l_{i_j})$, $1 \le j \le t$.

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Let

$$A_S[L, L'] = \prod_{\sigma \in \Sigma_t} [[A(L, l_{i_{\sigma(1)}}), A(L, l_{i_{\sigma(2)}})], \ldots, A(L, l_{i_{\sigma(t)}})]$$

be the symmetric (iterated) commutator subgroup of the normal subgroups of $A(L, l_{i_j})$ for $1 \le j \le t$. Clearly, $A_S[L, L'] \subset A(L, L')$, so $A(L, L')$ is always non-trivial.

Definition of Commutator quotient group

Consider the quotient group

$$\mathcal{A}(L, L') = \frac{A(L, L')}{A_S[L, L']}. \tag{3}$$

Definition

We call $\mathcal{A}(L, L')$ the commutator quotient group for a link pair (L, L').

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When $L' = L$, we call $\mathcal{A}(L, L)$ the commutator quotient group for a link L, and denote it by $\mathcal{A}(L)$ sometimes.

Definition of Commutator quotient group

Observe that the intersection subgroup (2) is given by the short exact sequence

$$A_S[L, L'] \rightarrowtail A(L, L') \twoheadrightarrow \mathcal{A}(L, L'),$$

a set of generators for $A_S[L, L']$ is understood in a obvious way.

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a set of generators for $A_S[L, L']$ is understood in a obvious way.

It is clear that the commutator quotient group $\mathcal{A}(L, L')$ for a link pair (L, L') is invariant under isotopy.

Links in 3-manifolds

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A link L in M is splittable if there exists a separating 2-sphere S in M, $S \cap L = \emptyset$, S cuts M into two pieces M_1' and M_2' such that $L \cap M_i' \neq \emptyset$ for $i = 1, 2$, or M_i' is not a 3-ball if $L \cap M_i' = \emptyset$ for $i = 1$ or 2.

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Let $L_1 = L \cap M_1'$ and $L_2 = L \cap M_2'$. Then $L = L_1 \sqcup L_2$ is a decomposition of the link L. Denote by M_i the 3-manifold obtained by capping off the resulting 2-sphere boundary component of M_i' with a 3-ball, $i = 1, 2$.

Links in 3-manifolds

Clearly, L is splittable if and only if M admits a nontrivial connected sum decomposition $M = M_1 \# M_2$ with $L_1 \subset M_1$, $L_2 \subset M_2$, and if $L_i = \emptyset$, M_i is not homeomorphic to the 3-sphere, for $i = 1$ or 2. Moreover, there is a connected sum decomposition of pairs as $(M, L) \cong (M_1, L_1) \# (M_2, L_2)$, and $M_L \cong M_{L_1} \# M_{L_2}$.

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We call L nonsplittable if L is not splittable. Clearly, L is nonsplittable if and only if M_L is prime.

Links in 3-manifolds

Remark

1. *The nontrivial connected sum $M = M_1 \# M_2$ means that both M_1 and M_2 are not S^3. Thus it is possible that L_1 the 0-link and M_2 is S^3 or L_2 the 0-link and M_1 is S^3. For instance, let L be a non-zero link in S^3, by taking the connected sum $S^3\#(S^2 \times S^1)$, we obtain a splittable link L in $S^3\#(S^2 \times S^1) \cong S^2 \times S^1$.*
2. *When $M = S^3$, this definition coincides with the usual definition of splittable links in S^3 because if $S^3_L \cong M_{L_1} \# M_{L_2}$ is a nontrivial connected sum, then both L_1 and L_2 are not a 0-link.*
3. *A 0-link is nonsplittable in M if and only if M itself is prime.*

Links in 3-manifolds

A pair of links (L, L_0) in a closed connected orientable 3-manifold M means a link L with a sublink L_0.

Definition

We call (L, L_0) strongly nonsplittable if any sublink L' of L with $L_0 \subsetneq L'$ is not splittable.

In other words, (L, L_0) is strongly nonsplittable if and only if any sublink L' of L which contains L_0 as a proper link is not splittable.

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A space X is called $K(\pi, 1)$ if $\pi_i(X) = 0$ for $i \geq 2$.

Links in 3-manifolds

In the following proposition we collect some facts about 3-manifolds:

Proposition

Let M be an orientable closed manifold with a link L in M.

1. *If $L = \emptyset$ is 0-link, then L is nonsplittable if and only if $M = S^3$, $S^2 \times S^1$ or irreducible.*
2. *If M is irreducible with $\pi_1(M)$ infinite, then M is a $K(\pi, 1)$-space.*
3. *If $L \neq \emptyset$, then L is nonsplittable if and only if M_L is irreducible.*
4. *If L is nonsplittable, then $\pi_2(M_L) = 0$.*
5. *If L is a nonempty nonsplittable link in M, then M_L is a $K(\pi, 1)$-space.*

Theorem (Fang-Lei-Wu)

Let (L, L_0) be a pair of links in a compact connected 3-manifold M such that $L \smallsetminus L_0$ is an n-link with $n \geq 2$ and (L, L_0) is strongly nonsplittable. Let L' be any sub t-link of $L \smallsetminus L_0$ with $2 \leq t \leq n$.

1. *If L' is a proper sublink of $L \smallsetminus L_0$, or L_0 is nonempty and nonsplittable, then $\mathcal{A}(L, L') = 0$.*

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1. *If L' is a proper sublink of $L \setminus L_0$, or L_0 is nonempty and nonsplittable, then $\mathcal{A}(L, L') = 0$.*
2. *If $L_0 = \emptyset$ and $L' = L$, then there is an isomorphism of groups*

$$\mathcal{A}(L) \cong \pi_n(M).$$

Remark. The first conclusion is equivalent to that the intersecting subgroup $A(L, L')$ is just the symmetric (iterated) commutator subgroup $A_S[L, L']$, and the latter has an obvious generator set.

Corollary

When the strongly nonsplittable link is lying in the 3-sphere, we have a direct consequence of the above Theorem as follow:

Theorem

Let L be an n-link in S^3 with $n \geq 2$. If L is strongly nonsplittable, then there is an isomorphism of groups

$$\mathcal{A}(L) \cong \pi_n(S^3).$$

Some Remarks

Remark

(1) How to understand the homotopy groups of the 3-sphere is a difficult question in algebraic topology. One of our motivations is to search certain "natural" and fundamental connections between homotopy groups and other areas of mathematics. Some progress has ever been made by Berrick-Cohen-Wong-Wu in 2006 for establishing connections between homotopy groups and Brunnian braids.

The above theorem gives a combinatoric description of the hommotopy groups of the 3-sphere in terms of link subgroups, which establishes a connection of the commutator quotient groups of intersecting subgroups of some link pair and the homotopy groups of the 3-sphere, which usually seem to be less related.

Some Remarks (continued)

Remark

(2) It is surprising that, though $A(L, L)$ is fully determined by the individual link L, the nth homotopy group $\pi_n(S^3)$ $(n \geq 2)$ only depends on the set of strongly nonsplittable link of n components, and is independent of the choice of individual links in the set, regardless to the difference of individual links in the set.

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(3) For a strongly nonsplittable n-links L in S^3, when $n = 3$, the group $\mathcal{A}(L) \cong \pi_3(S^3) = \mathbb{Z}$. For $n > 3$, the group $\mathcal{A}(L) \cong \pi_n(S^3)$ is a finite abelian group from the well-known results of Serre. For example, $\pi_4(S^3) = \mathbb{Z}/2$, $\pi_5(S^3) = \mathbb{Z}/2$, $\pi_5(S^3) = \mathbb{Z}/12$. Up to the range that the homotopy groups of S^3 are known (for instance, $n \leq 64$), the groups $\mathcal{A}(L_n; L_n)$ are fully determined by the above theorem.

Some Remarks (continued)

Remark

(4) On the other hand, the above theorem provides a "potential" way to determine the general homotopy groups $\pi_n(S^3)$, though generally speaking, it is a hopelessly difficult problem to fully determine the general homotopy groups $\pi_n(S^3)$ under current technology.

Outline Proof: Step 1

It was shown by Brown-Loday that there is a van Kampen-type theorem for determining the secondary homotopy groups as sub-quotients of certain fundamental groups through diagrams of spaces in 1987.

A generalization of Brown-Loday's theorem was given by Ellis-Steiner in 1987 and by Wu in 2010.

Here we have a more general version of Wu's result in the following:

Outline Proof: Step 1

Theorem

Let $(X; X_1, X_2, \ldots, X_n; X_0)$ be a cofibrant $K(\pi, 1)$ n-partition with $n \geq 2$. Suppose that the inclusion $X_0 \to X_i$ induces an epimorphism on the fundamental groups for each $1 \leq i \leq n$. Let R_i be the kernel of $\pi_1(X_0) \to \pi_1(X_i)$ for $1 \leq i \leq n$. Then

(i) *For any proper subset $I = \{i_1, \ldots, i_k\} \subsetneq \{1, 2, \ldots, n\}$,*

$$R_{i_1} \cap \cdots \cap R_{i_k} = [[R_{i_1}, R_{i_2}], \ldots, R_{i_k}]_S.$$

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(ii) *There is a natural isomorphism of $\mathbb{Z}[\pi_1(X_0)]$-modules*

$$\rho_{\mathbb{X}}\colon (R_1 \cap R_2 \cap \cdots \cap R_n)/[[R_1, R_2], \ldots, R_n]_S \xrightarrow{\cong} \pi_n(X)$$

where $\pi_1(X_0)$ acts on each R_i by conjugation and on $\pi_n(X)$ via the homomorphism $\pi_1(X_0) \to \pi_1(X)$ induced by the inclusion $X_0 \hookrightarrow X$. Moreover, the isomorphism is natural with respect to the morphisms of n-partitions.

Outline Proof: Step 2

Consider the homotopy theory on the cubical diagrams of spaces induced by link complements through deleting link components.

For strongly nonsplittable pair of links (L, L_0), the cubical diagram of the link complements obtained by removing the link components of $L \setminus L_0$ has the special property that the spaces in the cubical diagram are all $K(\pi, 1)$-spaces except the terminal space.

Then the above generalized van Kampen Theorem of can be applied to establish the connections between the fundamental groups and the higher homotopy groups of the terminal space in these special cases.

Brunnian-type knots

Let (L, L') be a link pair in S^3, where $L = \{l_1, l_2, \ldots, l_n\}$ and $L' = \{l_{i_1}, l_{i_2}, \ldots, l_{i_t}\}$. Recall $A(L, L') = \bigcap_{j=1}^{t} A(L, l_{i_j})$.

Brunnian-type knots

Let (L, L') be a link pair in S^3, where $L = \{l_1, l_2, \ldots, l_n\}$ and $L' = \{l_{i_1}, l_{i_2}, \ldots, l_{i_t}\}$. Recall $A(L, L') = \bigcap_{j=1}^{t} A(L, l_{i_j})$.

Given any element $\alpha \in A(L, L')$, one can choose a knot K in the link component $S^3 \smallsetminus L$ as a representative for the homotopy class α. The $(n+1)$-link $\tilde{L} = L \cup \{K\}$ admits the Brunnian-type property with respect to L', i.e., the knot K becomes a trivial knot (in homotopy sense) in the link complement $S^3 - d_{i_j} L$ for $1 \leq j \leq t$. That is, by removing any i_j th component of L, the knot K separates away from $d_{i_j} L$ and is homotopic to the trivial knot.

Brunnian-type knots

On the other hand, if a knot K in $S^3 \setminus L$ has the Brunnian-type property with respect to L', then $[K] \in A(L, L')$.

Thus the intersection subgroup $A(L, L')$ is corresponding to the Brunnian-type links.

Link invariants from $\pi_n(S_3)$

Recall that Milnor's μ-invariants for a Brunnian link L in S^3 are coming from $\mathcal{G}(L)$ (therefore $G(L)$).

For a strongly nonsplittable link L of n components, we have $\mathcal{A}(L) \cong \pi_n(S^3)$ $(n \geq 2)$.

If $\pi_n(S^3)$ is non-trivial, for $0 \neq \alpha \in \mathcal{A}(L)$, choose a knot K in the link component $S^3 \setminus L$ as a representative for the homotopy class α.

The $(n+1)$-link $\tilde{L} = L \cup \{K\}$ admits the Brunnian-type property with respect to L.

Clearly, α is an (homotopy) invariant for links $\tilde{L}$ (where L is fixed) or for the Brunnian type knot K with respect to L.

Link invariants from $\pi_n(S_3)$

We will show examples of $(n+1)$-links given in the form $\tilde{L} = L \cup \{K\}$ that have nontrivial homotopy-group invariants with the property that K represents the trivial element in Milnor's link group $\mathcal{G}(L)$.

In the other words, the Brunnian-type link $L \cup \{K\}$ is labeled by a nontrivial element in $\pi_n(S^3)$ (which means K is linked with L), but, as homotopy links in Milnor's sense, K is unlinked with L.

Examples

Consider the Hopf fibration $p\colon S^3 \longrightarrow S^2$.

Let $Q_n = \{q_1, \ldots, q_n\} \subseteq S^2$ be the n distinct points in S^2.

Let $L_n = p^{-1}(Q_n)$. Then $L_n = \{l_1, \ldots, l_n\}$ is an n-link in S^3, where $l_i = p^{-1}(q_i)$.

Let $I = \{i_1, \ldots, i_k\}$ be any nonempty subset of $\{1, \ldots, n\}$.

From the fibration

$$S^1 \longrightarrow S^3 \smallsetminus p^{-1}(q_{i_1}, \ldots, q_{i_k}) \longrightarrow S^2 \smallsetminus \{q_{i_1}, \ldots, q_{i_k}\},$$

we can see that the space $S^3 \smallsetminus p^{-1}(q_{i_1}, \ldots, q_{i_k})$ is a $K(\pi, 1)$-space.

Thus the sublink $\{l_{i_1}, \ldots, l_{i_k}\}$ is nonsplittable. It follows that L_n is a strongly nonsplittable n-link.

Examples

Theorem (Fang-Lei-Wu)

Let $\phi\colon G(L_n) \to \mathcal{G}(L_n)$ *be the quotient homomorphism for the Hopf link* L_n *with* $n \geq 4$. *Then*

$$A_1 \cap A_2 \cap \cdots \cap A_n \subseteq \mathrm{Ker}(\phi).$$

Examples

Theorem (Fang-Lei-Wu)

Let $\phi\colon G(L_n) \to \mathcal{G}(L_n)$ be the quotient homomorphism for the Hopf link L_n with $n \geq 4$. Then

$$A_1 \cap A_2 \cap \cdots \cap A_n \subseteq \mathrm{Ker}(\phi).$$

On the other hand, we know that $\mathcal{A}(L_n) \cong \pi_n(S^3)$ is non-trivial for many n. For example, $\pi_4(S^3) = \mathbb{Z}/2$, $\pi_5(S^3) = \mathbb{Z}/2$, $\pi_6(S^3) = \mathbb{Z}/12$, etc.

Thus for non-trivial $\pi_n(S^3)$ with $n \geq 4$, though Milnor's link group invariants (from $\mathcal{G}(L_n)$) vanish (so it gives less information for the Brunnian-type link $\tilde{L} = L_n \cup \{K\}$), but $\mathcal{A}(L_n, L_n) \cong \pi_n(S^3)$ works for $\tilde{L} = L_n \cup \{K\}$.

Thanks for your attention!