# § 3. Completely distinguishable projection

## Definition 3.1.

$\hat{f} : G \to \mathbb{R}^2$ reg. proj. with $p$ double pts.

$f_1, f_2, \dots, f_{2^p} : G \to \mathbb{R}^3$ sp. emb. obtained from $\hat{f}$

(1) $\hat{f}$ is completely distinguishable

$\overset{\text{def}}{\iff}$ $f_i \not\simeq f_j$ for $\forall i \neq j$

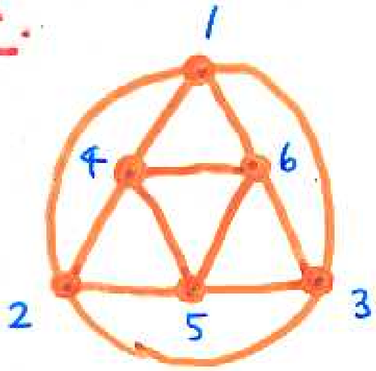
(2) A CDP $\hat{f}$ is trivial

$\overset{\text{def}}{\iff}$ $\hat{f}$ has no double pts.

## Remark.

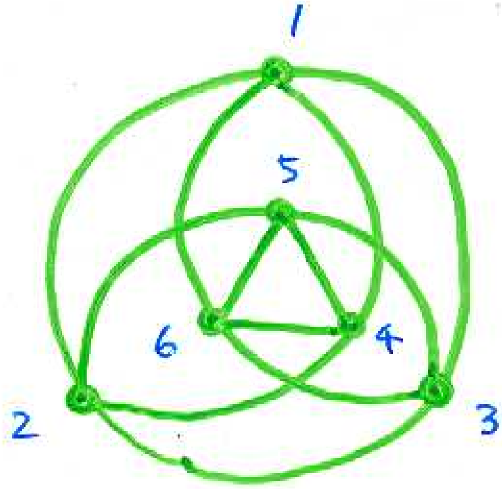
$\forall$ planar graph has a trivial CDP.

ex.

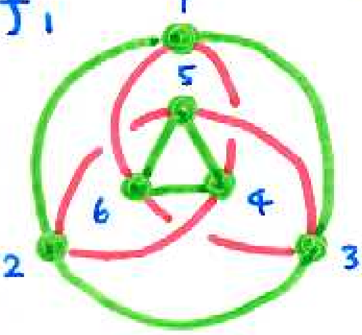



G

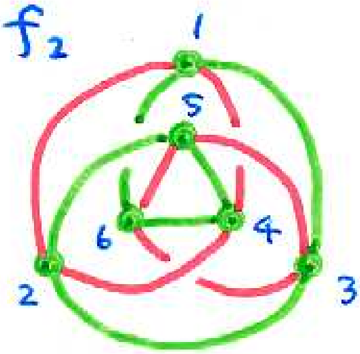
$\xrightarrow{\hat{f}}$



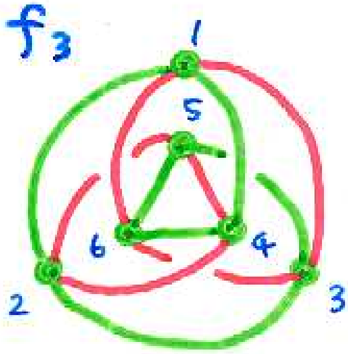

$f_1$



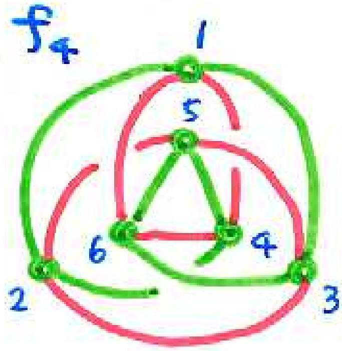
$f_2$



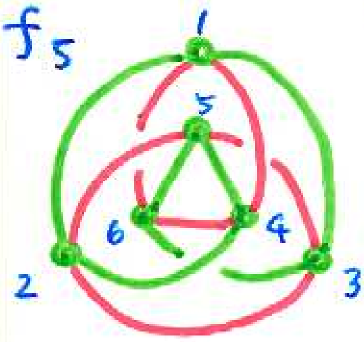
$f_3$



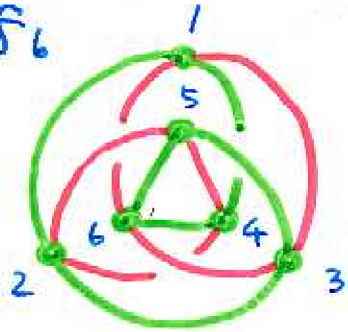
$f_4$



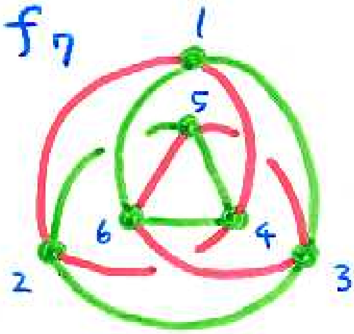
$f_5$



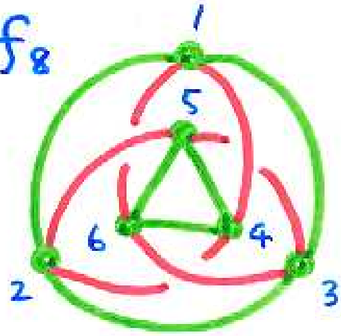
$f_6$



$f_7$



$f_8$



$\hat{f}$ is a non-trivial CDP.

## Proposition 3.2.

(1) $\begin{cases} G : \text{trivializable} \\ \hat{f} : G \to \mathbb{R}^2 \text{ reg. proj.} \end{cases}$

Then, $\hat{f}$ : CDP $\iff$ $\hat{f}$ has no double pts.

(2) $\begin{cases} G : \text{non-trivializable, planar} \\ \hat{f} : G \to \mathbb{R}^2 \text{ reg. proj.} \end{cases}$

Then, $\hat{f}$ : non-trivial CDP

$\Rightarrow$ $\hat{f}$ is knotted.

## Remark.

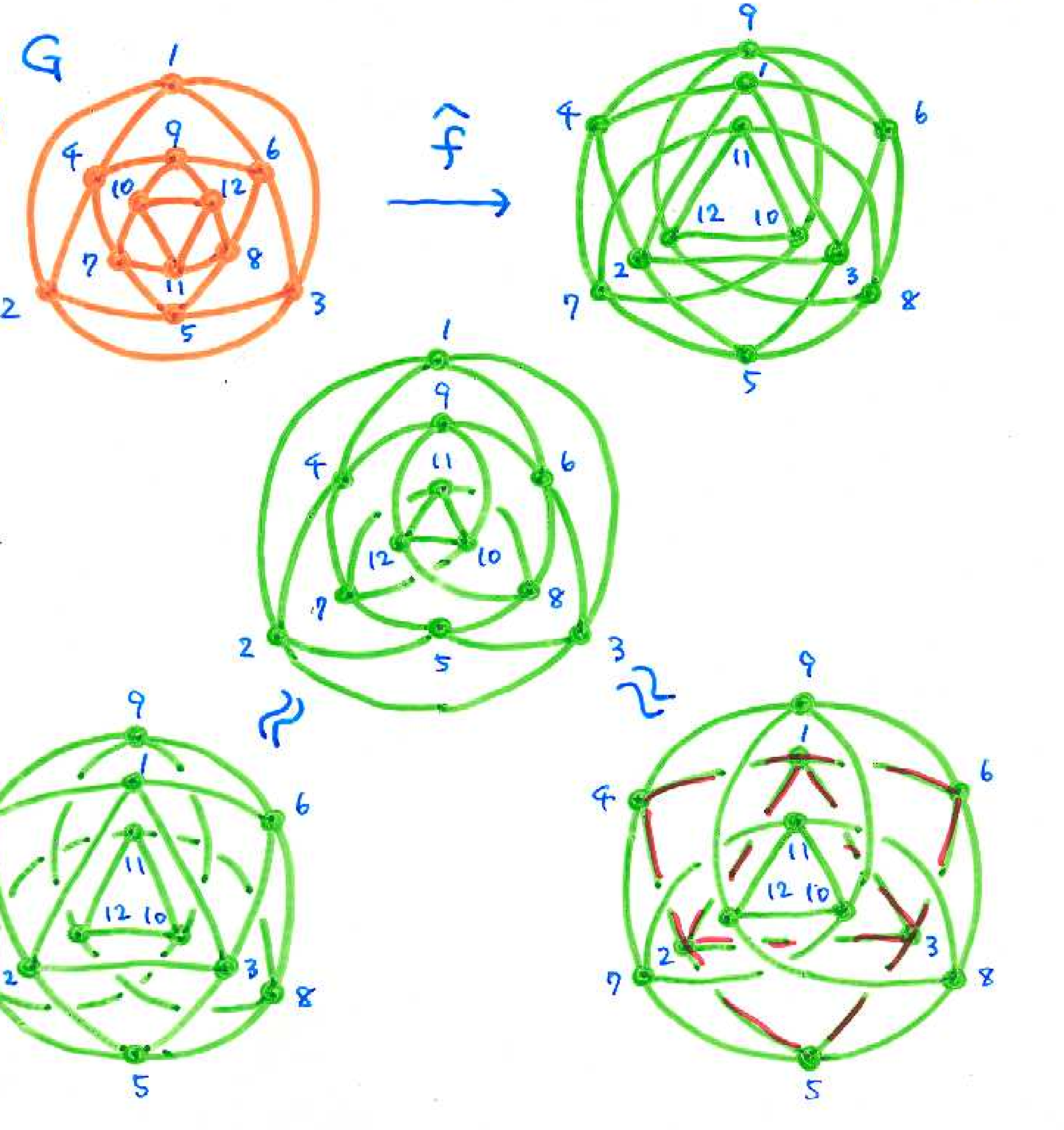
The converse of (2) is not true.

**ex.**

(1) By producing the local parts as

we can construct a KP which is not a CDP.

(2)



## Wu invariant :

$X$ : topological space

$$C_2(X) = \{ (x,y) \in X \times X \mid x \neq y \}$$

$$\begin{array}{ccccc} \sigma : & C_2(X) & \longrightarrow & C_2(X) & \\ & \Psi & & \Psi & \text{free involution.} \\ & (x,y) & \longmapsto & (y,x) & \end{array}$$

$$H^*(C_2(X), \sigma) \overset{\text{def}}{=} H^*(\operatorname{Ker}(1+\sigma_\#))$$

: skew-symmetric cohomology of $(C_2(X), \sigma)$

For sp. emb. $f : G \longrightarrow \mathbb{R}^3$,

$$\begin{array}{ccc} \mathbb{Z} \cong H^2(C_2(\mathbb{R}^3), \sigma) & \xrightarrow{(f^2)^*} & H^2(C_2(G), \sigma) \\ \Psi & & \Psi \\ \Sigma & \longmapsto & \mathcal{L}(f) \end{array}$$

generator

Wu invariant of $f$.

## Calculation :

$$C_2(X) \simeq D_2(X) = \bigcup_{\tau_1 \cap \tau_2 = \phi} \tau_1 \times \tau_2$$

$$(\ \tau_1, \tau_2 : \text{simplices of } X\ )$$

$$\begin{tikzcd}
H^2(C_2(\mathbb{R}^3), \sigma) \arrow[r, "(f^3)^*"] & H^2(C_2(G), \sigma) \arrow[d, "\cong"] \\
\mathbb{Z} \cong H^2(S^2, \sigma) \arrow[u, "r^*", "\cong"'] \arrow[r] \arrow[ur, phantom, "\circlearrowright"] & \boxed{H^2(D_2(G), \sigma)} \ni \mathcal{L}(f)
\end{tikzcd}$$

$$\left(\begin{array}{cccc} r : & C_2(\mathbb{R}^3) & \xrightarrow{\simeq} & S^2 \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (x, y) & \longmapsto & \dfrac{x-y}{\|x-y\|} \end{array}\right)$$

$$V(G) = \{ v_1, v_2, \cdots, v_m \}$$

$$E(G) = \{ e_1, e_2, \cdots, e_n \} \leftarrow$$ orientations are given

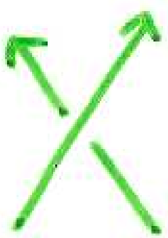
$$\begin{array}{c} 0 \\ \uparrow \\ C^2(D_2(G), \sigma) = \left\langle E^{ij} = (e_i \times e_j + e_j \times e_i)^{\#} \;\middle|\; e_i \cap e_j = \phi,\ 1 \leqq i < j \leqq n \right\rangle \\ \delta^1 \Big\uparrow \\ C^1(D_2(G), \sigma) = \left\langle V^{is} = (e_i \times v_s - v_s \times e_i)^{\#} \;\middle|\; e_i \cap v_s = \phi \right\rangle \end{array}$$

$$\boxed{\delta^1(V^{is}) = \sum_{I(j)=s} E^{P(ij)} - \sum_{T(k)=s} E^{P(ik)}}$$

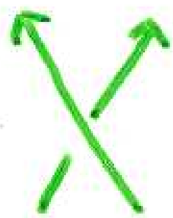
$$P(ij) = \begin{cases} ij & (i < j) \\ ji & (i > j) \end{cases}$$

$$I(j) = s \overset{\text{def}}{\Longleftrightarrow} v_s \xrightarrow{e_j} \bullet, \qquad T(k) = s \overset{\text{def}}{\Longleftrightarrow} \bullet \xrightarrow{e_k} v_s$$

$p$ :



$\varepsilon(p) = 1$



$\varepsilon(p) = -1$

$$a_{ij}(f) \overset{\text{def}}{=} \sum_{p \in \pi \circ f(e_i) \cap \pi \circ f(e_j)} \varepsilon(p)$$

$$\boxed{\mathcal{L}(f) = \Big[ \sum_{\substack{e_i \cap e_j = \phi \\ 1 \leq i < j \leq n}} a_{ij}(f)\, E^{ij} \Big] \in H^2(D_2(G), \sigma)}$$

**Remark.**

(1) $H^2(D_2(G), \sigma)$ is torsion free.

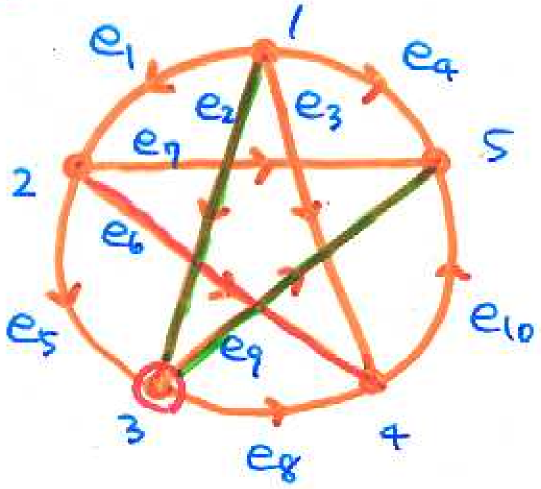
(2) $\mathcal{L}(f) = -\mathcal{L}(f!)$

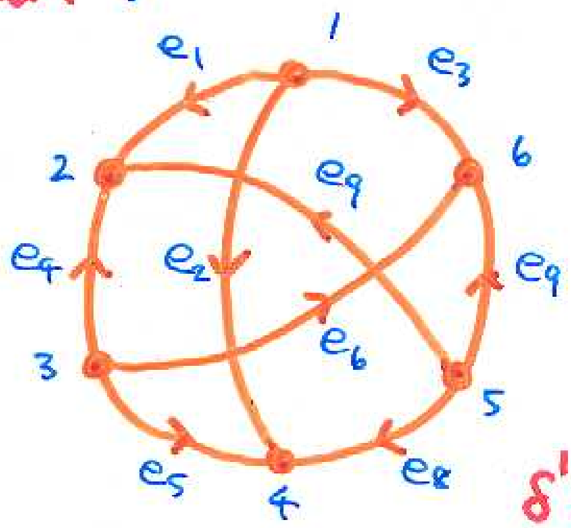
(3) $\mathcal{L}(f)$ coincides with the **Simon invariant** if $G \approx K_5$ or $K_{3,3}$

(4) $\mathcal{L}(f) = 2\,\mathrm{lk}(f)$

if $G \approx S^1 \sqcup S^1$.

**ex.** ( Simon invariant )



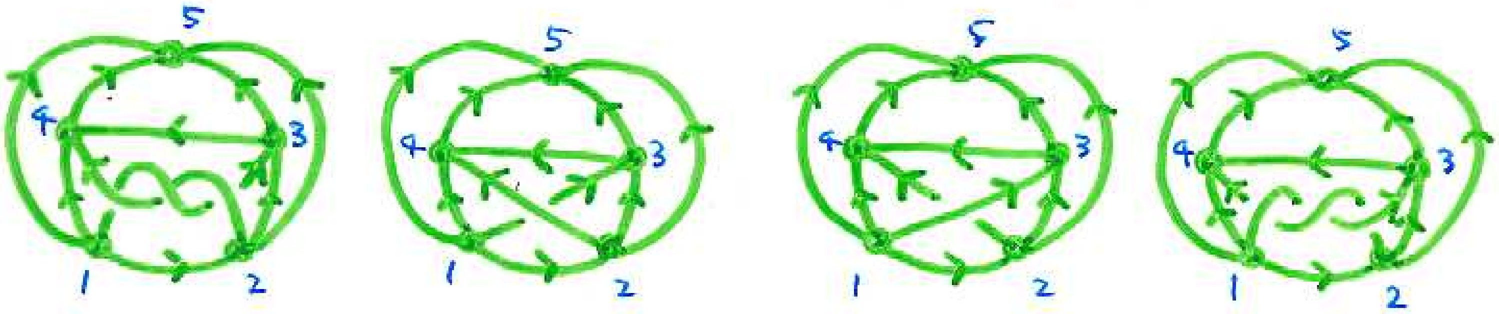




$$H^2(D_2(K_5),\sigma) = \langle E^{26}, E^{69}, \cdots \mid \underbrace{-E^{26}+E^{69}}_{}, \cdots \rangle \qquad \delta'(v^{63})$$

$$= \langle E^{26} \rangle \cong \mathbb{Z}$$

$$H^2(D_2(K_{3,3}),\sigma) = \langle E^{27}, E^{24}, \cdots \mid \underbrace{-E^{27}-E^{24}}_{}, \cdots \rangle \qquad \delta'(v^{22})$$

$$= \langle E^{27} \rangle \cong \mathbb{Z}$$



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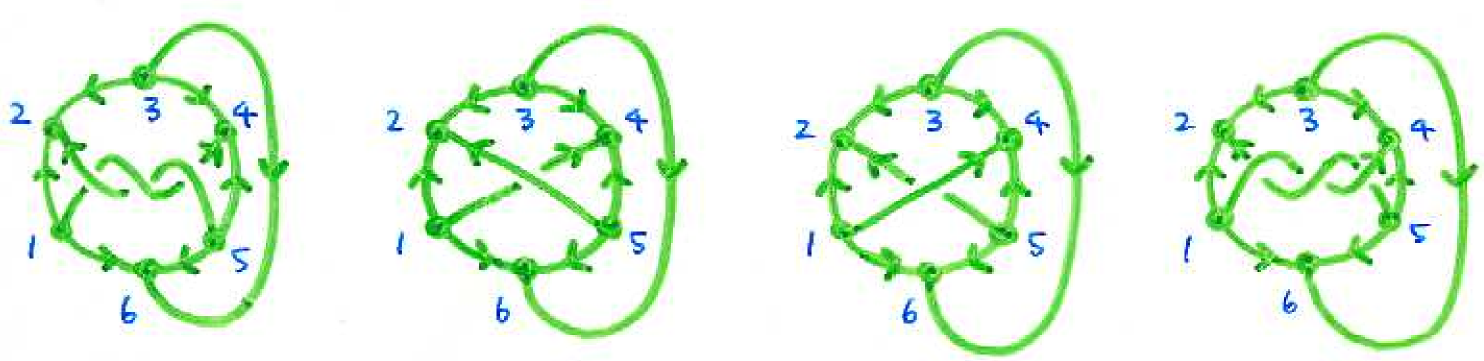


## Theorem 3.3.

$K_n$ and $K_{m,n}$ have a CDP.

## Remark.

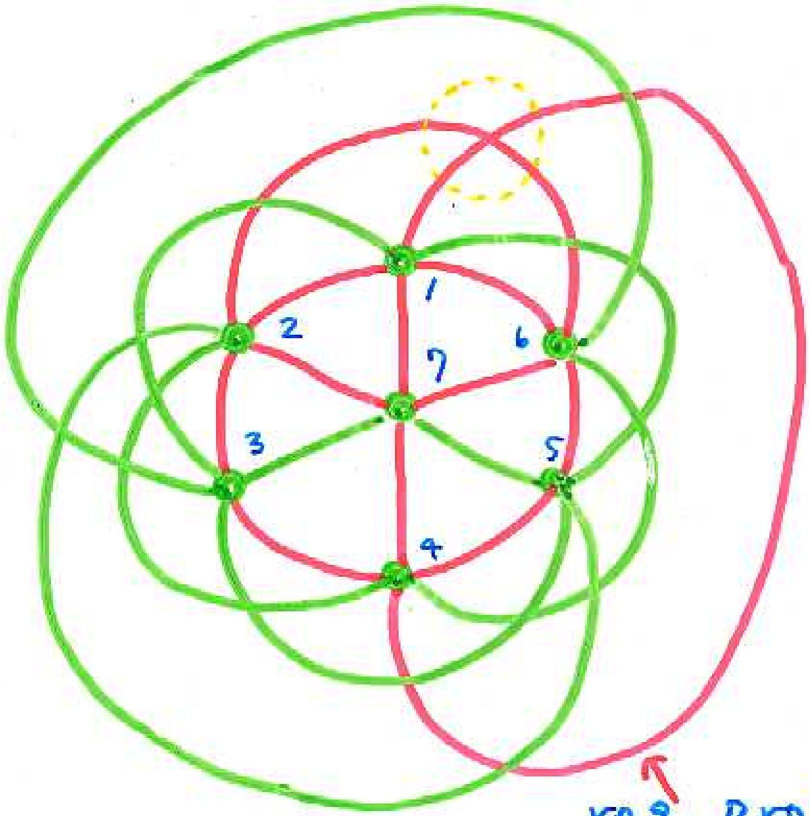
(1) $K_n$ is trivializable if $n \leqq 4$.

(2) $K_{m,n}$ is trivializable if $\min\{m, n\} \leqq 2$.

In the case above, only embeddings into $\mathbb{R}^2$ are CDP. ($\because$ Prop. 3.2 (1))

## ex. ( The case of $K_7$ )

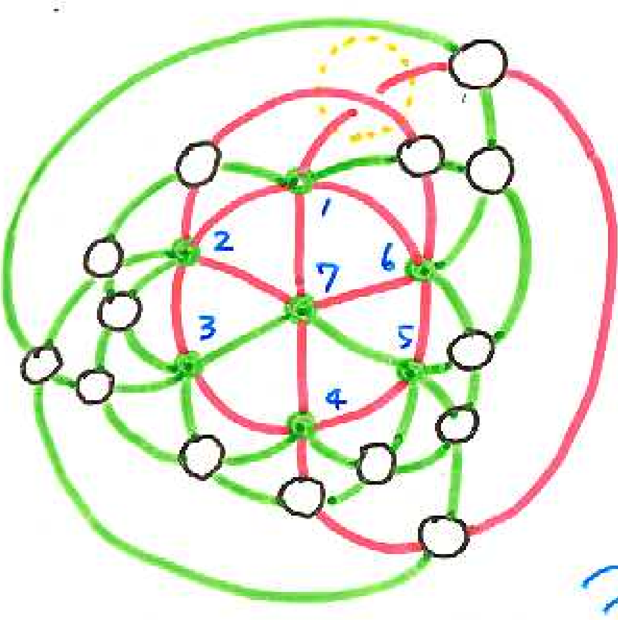
$K_7 \xrightarrow{\hat{f}}$



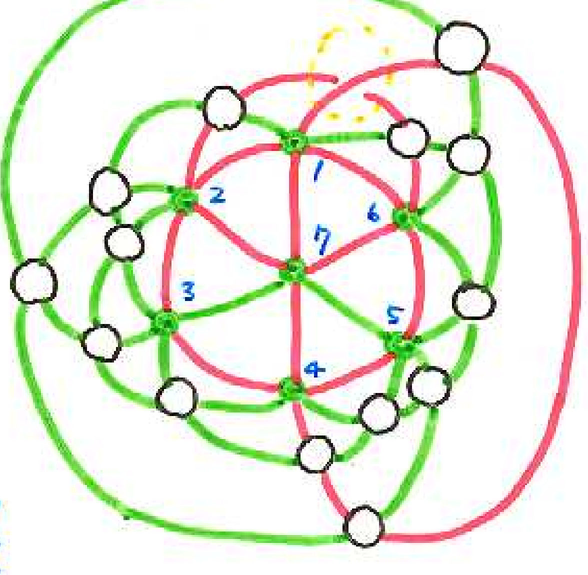

reg. proj. ~~spatial~~

$K_5$ - graph

$\pi$ $\nearrow$ $\quad$ $\uparrow$ $\pi$




$\not\simeq$



# Minimal crossing projection

$\hat{f} : G \to \mathbb{R}^2$ reg. proj.

$c(\hat{f}) \overset{\text{def}}{=} \#\{\text{double pts. of } \hat{f}\}$

## Definition 3.5.

(1) $c(G) \overset{\text{def}}{=} \min\{c(\hat{f}) \mid \hat{f} : \text{reg. proj. of } G\}$

: minimal crossing number of $G$.

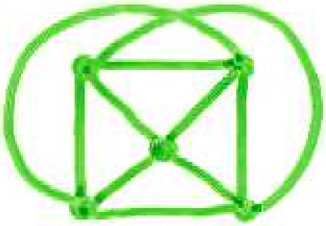
(2) $\hat{f} : G \to \mathbb{R}^2$ reg. proj.

is a minimal crossing projection

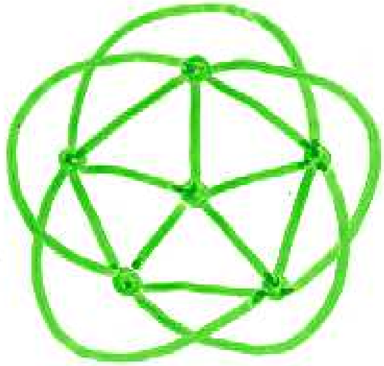
$\overset{\text{def}}{\Longleftrightarrow} \quad c(\hat{f}) = c(G).$

## Remark.

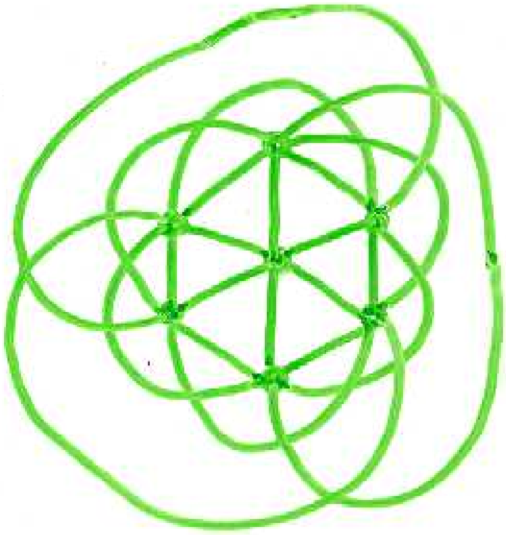
If $G$ is planar, then $c(G) = 0$.



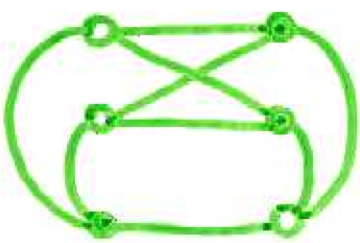
$K_5$



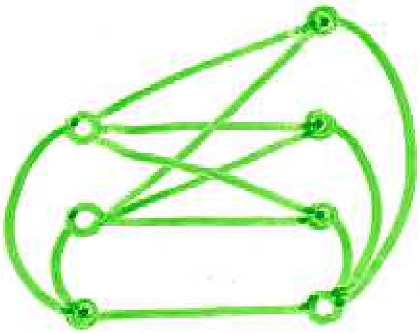
$K_6$



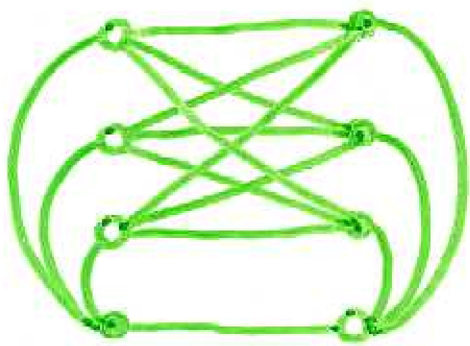
$K_7$ ....



$K_{3,3}$



$K_{3,4}$

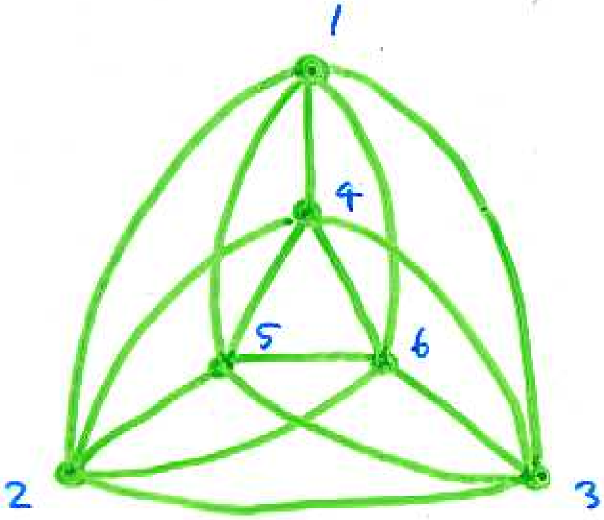


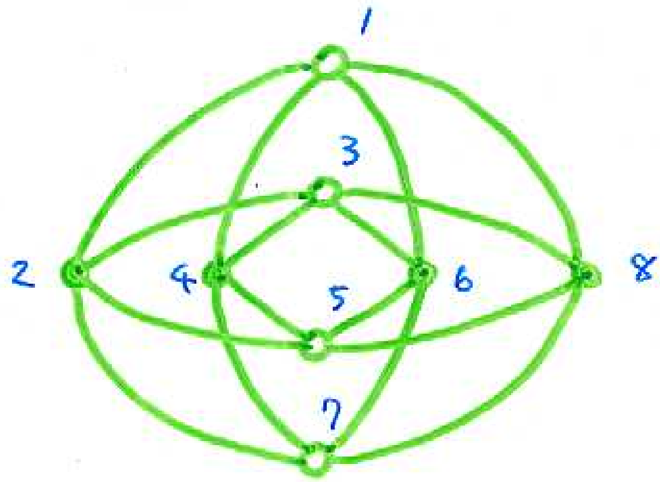
$K_{4,4}$ ...

**Conjecture 3.4.**

Every graph has a CDP.

**ex.** $c(K_6) = 3$, $c(K_{4,4}) = 4$.



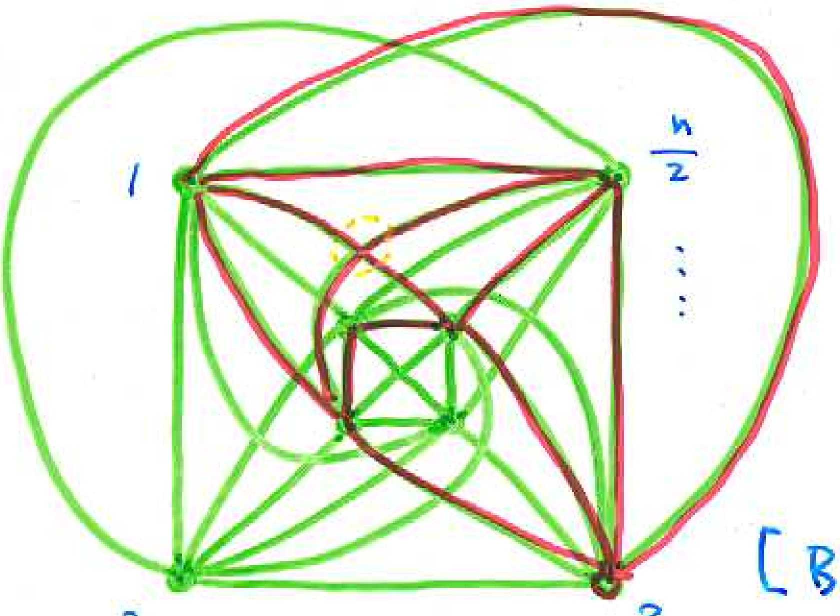

**Question.** (asked by Ozawa)

Is a minimal crossing projection a CDP ?

$\Rightarrow$ If $c(G) \leq 1$, Yes.

# Candidates for min. cr. proj. of $K_n$ and $K_{m,n}$

(1)



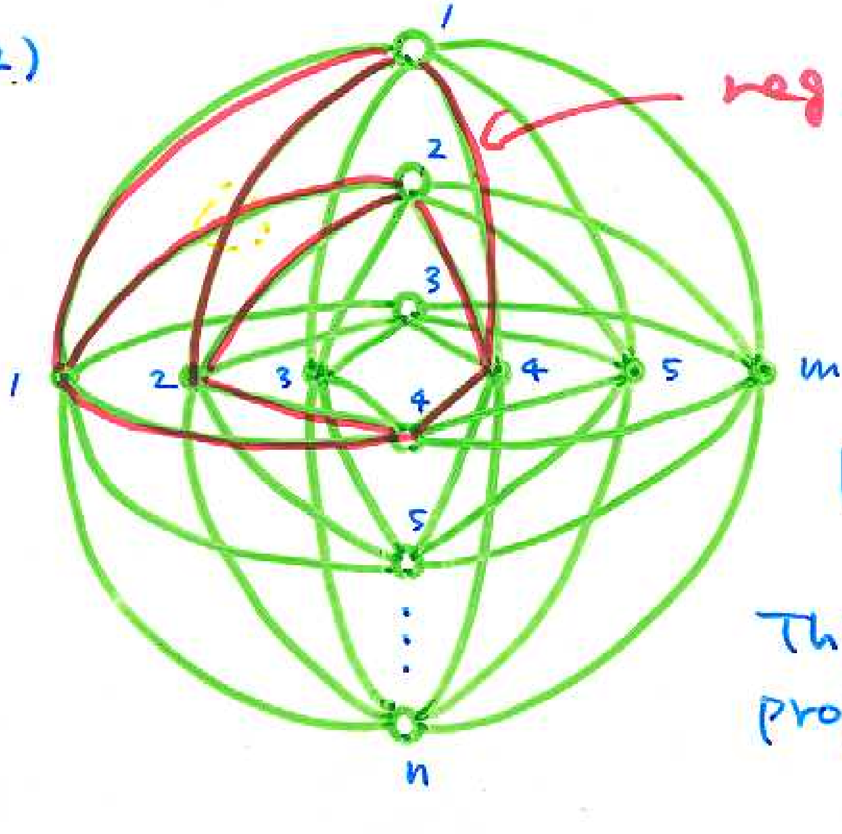

← reg. proj. of $K_5$

$n$ : even

[Blažek - Koman]

This is a min. cr. proj if $n = 6, 8, 10$.

(2)




reg. proj. of $K_{3,3}$

[Zarankiewicz]

This is a min. cr. proj. if $m \leqq 6$.
[Kleitman]

## Conjecture 3.6.

Every minimal crossing projection is a CDP.