Slide 1.

The $\mathrm{SL}(2,\mathbb{C})$ character variety of the one-holed torus.

Ser Peow Tan, Yan Loi Wong,
and Ying Zhang

Department of Mathematics
National University of Singapore
Preprint: http://front.math.ucdavis.edu/math.GT/0502464

OVERVIEW

$\mathbb{T}$ one-holed torus, $\Gamma = \pi_1(\mathbb{T})$.

Study the $\mathrm{SL}(2,\mathbb{C})$ character variety of T

$$\mathcal{X} = Hom(\Gamma, \mathrm{SL}(2,\mathbb{C}))//\mathrm{SL}(2,\mathbb{C}),$$

the *large scale behavior* of the action of $\mathcal{MCG}$ on $\mathcal{X}$, *via*

the *(generalized) Markoff maps* $\phi \in \mathbf{\Phi}$.

Applications: "moduli space", Generalized McShane's identity, variations, punctured torus bundles over circle, closed hyperbolic 3-manifolds.

Key: (i) Bowditch's Q-conditions, (ii) Markoff maps encode all information.

Notation and definitions

- $\Gamma = \langle X, Y \rangle = \pi_1(\mathbb{T})$, where $\mathbb{T}$ is the one-hole torus.
- $\rho \in Hom(\Gamma, \mathrm{SL}(2,\mathbb{C}))$ is a τ representation if $tr\rho[X,Y] = \tau$.
- $\mathcal{X}$, ($\mathcal{X}_\tau$): the (relative) character variety.
- $\mathcal{C} = \{$ essential simple closed curves on $\mathbb{T}\}$
- $\mathcal{MCG} = \pi_0(\mathrm{Homeo}(\mathbb{T}))$ acts transitively on $\mathcal{C}$.
- **For** $\gamma \in \mathcal{C}$**,** $l(\rho(\gamma)) \in \mathbb{C}/2\pi i\mathbb{Z}$ (complex length) :

$$l(\gamma) := l(\rho(\gamma)) = 2\cosh^{-1}(\frac{tr\ A}{2})$$

where $A = \rho(\gamma) \in \mathrm{SL}(2,\mathbb{C})$.

Motivation

Theorem 0.1 *(Greg McShane) McShane's Identity*
If $\rho : \Gamma \to \mathrm{SL}(2, \mathbb{R})$ *is the holonomy representation of a complete, finite area hyperbolic structure on the punctured torus (*$\tau = -2$*). Then*

$$\sum_{\gamma \in \mathcal{C}} \frac{1}{1 + e^{l(\rho(\gamma))}} = \frac{1}{2}, \tag{1}$$

where the sum converges absolutely.

Generalizations:

(1) (McShane) Hyperbolic surfaces with cusps, Weierstrass identities

(2) (Bowditch) quasi-fuchsian representations, with independent proof via Markoff triples. Punctured torus bundles

(3) (Akiyoshi-Miyachi-Sakuma) Refinements of Bowditch, punctured surface bundles

(4) (Mirzakhani) Surfaces with geodesic boundary, applications to Weil-Petersen volume, etc.

(5) (T-Wong-Zhang) Cone surfaces with cusps/geodesic boundary, Schottky groups, genus two surface.

(6) This talk: combine (2) and (5), generalizations to $\mathrm{SL}(2,\mathbb{C})$ characters of the one-holed torus

Theorem 0.2 *(T., Wong, Zhang)*

Suppose $\rho \in \mathcal{X}_\tau : \Gamma \to \mathrm{SL}(2,\mathbb{C})$ *satisfies the Bowditch Q-conditions as follows:*

1. $tr\ \rho(\gamma) \notin [-2,2]$ *for all* $\gamma \in \mathcal{C}$ *(no elliptics or accidental parabolics for simple closed curves).*
2. $|tr\ \rho(\gamma)| \leq 2$ *for only finitely many (possibly no)* $\gamma \in \mathcal{C}$.

Then

$$\sum_{\gamma \in \mathcal{C}} \log \frac{e^{\nu} + e^{l(\rho(\gamma))}}{e^{-\nu} + e^{l(\rho(\gamma))}} = \nu \mod 2\pi i, \tag{2}$$

where $\nu = \cosh^{-1}(-\tau/2)$,

and the sum converges absolutely.

Remarks.

- $\nu = \frac{1}{2}l([\rho(a), \rho(b)])$ (one of two choices).
- Also true for representations into $\mathrm{PSL}(2, \mathbf{C})$.
- Many geometric examples satisfy BQ-conditions. E.g. one-hole or one-cone hyperbolic torus, hyperbolic three holed sphere, quasi-fuchsian structures, classical Schottky structures.
- BQ-conditions are never satisfied when $\tau = 2$.
- When $\tau = -2$, the identity (2) is trivial, but the derivative with respect to ν evaluated at $\nu = 0$ gives McShane's original identity (1) as well as Bowditch's generalization.

Slide 8.

Theorem 0.3 *(T. Wong, Zhang).* *If we replace condition (1) in Theorem 0.2 by*

(1') $tr\ \rho(\gamma) \notin (-2,2)$ for all $\gamma \in \mathcal{C}$ (accidental parabolics are allowed),

then (1') and (2) are necessary and sufficient conditions for the identity (2) to hold.

Slide 9.

$\mathcal{M}CG$ acts on $\mathcal{X}$ and $\mathcal{X}_\tau$ by:

$$H(\rho)(\gamma) = \rho(H_*^{-1}(\gamma))$$

where $H \in \mathcal{M}CG$, H_* is the induced action on Γ.
$\mathcal{M}CG$ acts on the BQ-subspace. We have:

Theorem 0.4 *(Bowditch, T- Wong-Zhang). (i) The BQ-subspace is open in the space of τ-representations.*
(ii) $\mathcal{M}CG$ acts properly discontinuously on the BQ-subspace. Moreover this is the largest open subspace for which the action is properly discontinuous.

Theorem 0.5 *(T., W., Z.)* *Suppose* $\rho \in \mathcal{X}_\tau$ *is stabilized by a hyperbolic* $H \in \mathcal{MCG}$, *i.e.* $H(\rho) = \rho$. *Suppose further that* $\mathcal{C}/\langle H\rangle$ *satisfies the BQ-conditions. Set* $\nu = \cosh^{-1}(-\tau/2)$. *Then*

$$\sum_{\gamma\in\mathcal{C}/\langle H\rangle} \log \frac{e^{\nu} + e^{l(\gamma)}}{e^{-\nu} + e^{l(\gamma)}} = 0 \mod 2\pi i, \tag{3}$$

where the sum converges absolutely.

Theorem 0.6 *(T., W., Z.)* $\rho : \Gamma \to \mathrm{SL}(2,\mathbb{C})$ *as above. Then*

$$\sum_{\gamma\in\mathcal{C}_L/\langle H\rangle} \log \frac{e^{\nu} + e^{l(\gamma)}}{e^{-\nu} + e^{l(\gamma)}} = \pm\lambda \mod 2\pi i, \tag{4}$$

where $\mathcal{C} = \mathcal{C}_L \sqcup \mathcal{C}_R$ *is partition induced by* H, λ *is the complex length of the conjugating transformation in* $\mathrm{PSL}(2,\mathbb{C})$ *corresponding to* H.

Remark: Theorems 0.5 and 0.6 can be stated in terms of incomplete hyperbolic structures on punctured torus bundles over the circle.

Natural correspondence

- Φ_μ: μ-Markoff maps.
- $\mathcal{X}_{\mu-2}$: The $(\mu-2)$-relative character variety.
- $\mathcal{V}_\mu = \{(x,y,z) \in \mathbb{C}^3 \mid x^2+y^2+z^2-xyz=\mu\}$:

$\mathcal{X}_{\mu-2} \longleftrightarrow \mathcal{V}_\mu$:

Fix generators X, Y for Γ, then
$\rho \longrightarrow (tr\rho(X), tr\rho(Y), tr\rho(XY))$. (Fricke trace identity).

For $(x,y,z) \in \mathcal{V}_\mu$, let

$$\rho(X) = A := \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}, \rho(Y) = B := \begin{pmatrix} 0 & \zeta \\ -\zeta^{-1} & y \end{pmatrix},$$

then $\rho(XY) = AB := \begin{pmatrix} \zeta^{-1} & * \\ * & \zeta \end{pmatrix}$, where $z = \zeta + \zeta^{-1}$.

Definition of μ-Markoff maps.

- $\mathbb{H}^2$ - the hyperbolic plane
- $\mathcal{F}$ - the Farey tessellation of $\mathbb{H}^2$.
- Σ - binary tree dual to $\mathcal{F}$.
- complementary region - Connected component of $\mathbb{H}^2 \setminus \Sigma :$, use X, Y, Z to denote.
- $\{Comp.\ \ regions\ \ of\ \ \Sigma\} \leftrightarrow \mathbb{Q} \cup \infty \leftrightarrow \mathcal{C}$
- $V(\Sigma), E(\Sigma), \Omega \longleftrightarrow \{vertices\}, \{edges\}, \{comp.\ \ regions\}$.
- For $e \in E(\Sigma)$, use $e \leftrightarrow (X, Y; Z, W)$ if $X \cap Y = e$, and Z, W meet e at the ends of e.

A *μ-Markoff map* is a function $\phi : \Omega \to \mathbb{C}$ such that

(i) for all vertices $v \in V(\Sigma)$,

$$x^2 + y^2 + z^2 - xyz = \mu, \tag{5}$$

where $x = \phi(X), y = \phi(Y), z = \phi(Z)$, and $X, Y, Z \in \Omega$ are the three regions meeting v; and

(ii) if $e \in E(\Sigma)$, we have

$$xy = z + w, \tag{6}$$

where $e \leftrightarrow (X, Y; Z, W)$,
$x = \phi(X), y = \phi(Y), z = \phi(Z), w = \phi(W)$.

By (6), (5) holds for all vertices if it holds for one.

Correspondence between Φ_μ and $\mathcal{V}_\mu$:

Fix $X, Y, Z \in \Omega$ meeting at a vertex $v \in V(\Sigma)$.

Map from Φ_μ to $\mathcal{V}_\mu$:

$\phi \mapsto (\phi(X), \phi(Y), \phi(Z))$.

Inverse map generated by edge relations (6) of ϕ.

Note: $\Omega \equiv \mathcal{C} \equiv \mathbb{Q} \cup \infty$.

Action of $\mathrm{PSL}(2, \mathbb{Z})$ on Φ_μ:

$\mathrm{PSL}(2, \mathbb{Z})$ acts on Σ, hence Ω. If $g \in \mathrm{PSL}(2, \mathbb{Z})$, $g(\phi)$ is given by

$$g(\phi)(X) = \phi(g^{-1}(X)), \quad \forall X \in \Omega.$$

This corresponds to the action of $\mathcal{MCG}$ on $\mathcal{X}_{\mu-2}$, and $\mathcal{V}_\mu$.

Examples

Example 1. $\phi \in \mathbf{\Phi}_0 \leftrightarrow (3,3,3)$, (Original Markoff triples). Satisfies BQ-conditions ($\Omega(2) = \emptyset$). T is one vertex (sink).

Example 2. $\phi \in \mathbf{\Phi}_\mu \leftrightarrow (3+\epsilon_1, 3+\epsilon_2, 3+\epsilon_3)$, $\epsilon_1, \epsilon_2, \epsilon_3$ small.

Example 3. $\phi \in \mathbf{\Phi}_0 \leftrightarrow (\frac{3+\sqrt{-3}}{2}, \frac{3-\sqrt{-3}}{2}, \frac{3-\sqrt{-3}}{2})$. ϕ invariant under H, a conjugate of $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Corresponds to complete finite volume hyperbolic structure on the figure eight knot complement, $\Omega/\langle H \rangle$ satisfies BQ-conditions.

Example 4. $\phi \in \mathbf{\Phi}_\mu$ corresponding to small deformation of example 3 invariant under H.

BQ-conditions for μ-Markoff maps:

$\phi \in \Phi_\mu$ satisfies the extended BQ-conditions if:

1. $\phi(X) \notin (-2, 2)$ for all $X \in \Omega$.
2. $|\phi(X)| \leq 2$ for only finitely many $X \in \Omega$.

The results (Theorems 0.2-0.6) are converted to equivalent versions involving μ-Markoff maps and proven.

Results for μ-Markoff maps

Fix $\phi \in \mathbf{\Phi}_\mu$ once and for all.

Definition: $\Omega(k) := \Omega_\phi(k) = \{X \in \Omega \ : \ |\phi(X)| \le k\}$.

Lemma 0.7 *(Bowditch, T.W.Z) For all $k \ge 2$, $\Omega(k)$ is connected as a subset of $\mathbb{H}^2$. In particular, $\Omega(2)$ is connected.*

Arrows assigned by ϕ:

For each edge $e \leftrightarrow (X, Y; Z, W) \in E(\Sigma)$, we assign an arrow pointing from W to Z if $|\phi(W)| \ge |\phi(Z)|$.

Theorem 0.8 *(Bowditch, T.,W.,Z.) If $\phi \in \mathbf{\Phi}_\mu$ satisfies the BQ-conditions, then there exists a finite subtree $T \subset \Sigma$ (attractor) such that the arrows on all edges of Σ not in T points towards T.*

Growth rates of μ-Markoff maps

Fibonacci Function F_e:

For a fixed $e \in E(\Sigma)$, the Fibonacci function

$$F_e : \Omega \mapsto \mathbb{N}$$

defined inductively by :
(1) $F_e(X) = F_e(Y) = 1$ for the two regions X, Y adjacent to e;
(2) if X, Y, Z are three regions meeting at a vertex such that $F_e(X)$ and $F_e(Y)$ are already defined, then $F_e(Z) = F_e(X) + F_e(Y)$.

A function $f : \Omega \mapsto \mathbb{R}_+$ has Fibonacci growth if there exists $k > 0$ such that

$$k^{-1} F_e(X) \leq f(X) \leq k F_e(X),$$

for all (except a finite number) of $X \in \Omega$.

Remark: It does not matter which edge e we use.

Theorem 0.9 *(Bowditch, T.,W.,Z.) If* ϕ *satisfies the BQ-conditions, then* $\log^+|\phi|$ *has Fibonacci growth, where* $\log^+|\phi|(X) = \max\{0, \log|\phi(X)|\}$.

In particular, the lower Fibonacci bound implies

$$\sum_{\gamma \in \mathcal{C}} \log \frac{e^{\nu} + e^{l(\rho(\gamma))}}{e^{-\nu} + e^{l(\rho(\gamma))}}$$

converges absolutely.

The edge weight $\psi(\vec{e})$.

There exists a function $\psi_\phi = \psi : \vec{E} \mapsto \mathbb{C}$ satisfying

(1) $\psi(\vec{e}) + \psi(-\vec{e}) = \nu$

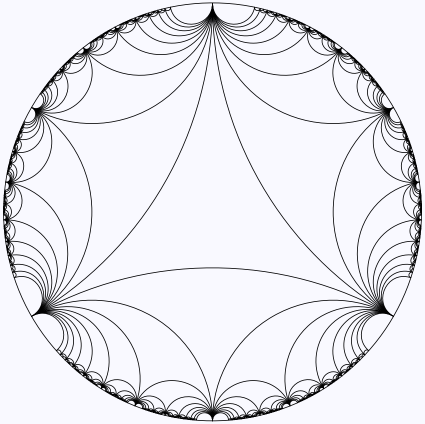
(2) $\psi(\vec{e}_1) + \psi(\vec{e}_2) + \psi(\vec{e}_3) = \nu$, whenever $\vec{e}_1, \vec{e}_2, \vec{e}_3$ all point into the same vertex v.

Using this function, and the limits of these functions, we can evaluate the value of the infinite sums, and prove Theorems 0.2-0.6.

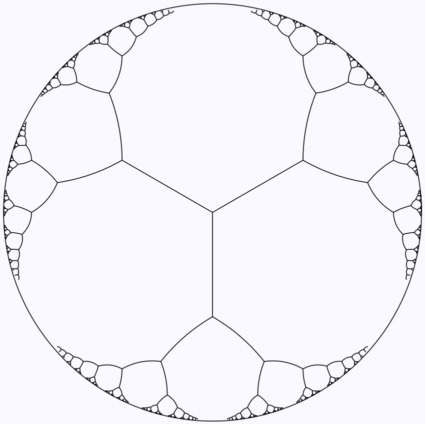
Summary and Speculations:

- The generalized Markoff maps provide a very effective way to study the space of representations of Γ to $\mathrm{SL}(2, \mathbb{C})$.
- McShane's identity, and various generalizations and variations can be proved by analysing corresponding gen. Markoff maps.
- The BQ-conditions are central. They "replace" the discrete and faithful and geometrically finite conditions in the type-preserving case. The subspace of representations satisfying the BQ-conditions is natural and important.
- Various analogues of Kleinian group theory type results can be formulated/conjectured for general representations (for example the ending lamination theorem, rigidity type results.

THE END



The Farey Tessellation $\mathcal{F}$.



The Dual Tree Σ.