Unknotting theorem
for
delta and sharp
edge-homotopy

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§1. Introduction

Crossing change:

Delta move:

Sharp move:

Each of them is known as an unknotting operation.

\[ \Delta : [ \text{H. Murakami - Nakanishi} ] , \]

\[ \# : [ \text{H. Murakami} ] [ \text{Matveev} ] \]
Definition 1.1. [Taniyama]

\[ f, g : G \to S^3 \text{ sp. emb.} \]

\( f \) and \( g \) are edge-homotopic if
\[ \iff f \text{ and } g \text{ are transformed into each other by self crossing changes and ambient isotopies.} \]

Here, self crossing change
\[ = \text{ crossing change on the same spatial edge} \]

This is a generalization of Milnor's link homotopy.

\[ \text{EH} \sim \]

\[ \approx \]
Definition 1.2. \([N]\)

(1) \(f, g : G \rightarrow S^3\) sp. emb.

\(f\) and \(g\) are delta edge-homotopic

\(\iff\) \(f\) and \(g\) are transformed into each other by self delta moves and ambient isotopies.

This is a generalization of self \(\Delta\)-equivalence of links [Shibuya].
(2) $G$: oriented graph
$f, g : G \to S^3$ sp. emb.

$f$ and $g$ are sharp edge-homotopic

def \[ \iff \] $f$ and $g$ are transformed into each other by self sharp moves and ambient isotopies.

This is a generalization of self $\#$-equivalence on links [Shibuya].
(1) Sharp edge-homotopy does not depend on the edge-orientations.

If we turn the orientations of all strings in a sharp move the other way at once, then the concluded move is also a sharp move.

(2) (DEH) $\Rightarrow$ (#EH) $\Rightarrow$ (EH)

A delta move can be realized by sharp moves

[Murakami - Nakanishi]
Theorem 1.3. [Taniyama]

For a graph $G$, the following are equivalent.

1. \exists f, g : G \rightarrow S^3 \text{ sp. emb. are edge-homotopic.}

2. $G / \equiv H \cong \begin{array}{c} \text{or} \\ K_4 \end{array}$ or $\begin{array}{c} \text{or} \\ D_3 \end{array}$

3. $G$ is a generalized bouquet, namely $\exists v : \text{vertex of } G$

such that $H_s, t, H(t) (G - v \backslash \&) = 0$.

ex.
Main Theorem.

For a graph \( G \) which does not have a free vertex, the following are equivalent.

(1) \( \forall f, g : G \rightarrow S^3 \) sp. emb. are delta edge-homotopic,

(2) \( \forall f, g : G \rightarrow S^3 \) sp. emb. are sharp edge-homotopic.

(3) \( G \not\cong \bigcirc \lor \bigcirc \bigcirc \)

(4) \( G \) is a bouquet, namely \( \exists m \in \mathbb{N} \)

s.t. \( G \cong \ast \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \)

where \( m \) is a natural number.
Remark.

(1) All of 2-comp. links have been classified completely up to

\[
\begin{align*}
\text{DEH} & : \ [\text{Nakanishi-Ohyama}] \\
\#\text{EH} & : \ [\text{Shibuya}] \\
\end{align*}
\]

1k and generalized Sato-Levine Invariant

1k and reduced Art invariant

(2) All of spatial \(\Theta\)-curves have been classified completely up to

\[
\begin{align*}
\text{DEH} & : \ [N] \\
\#\text{EH} & : \ [N] \\
\end{align*}
\]

Sato-Levine invariant of the associated 2-comp. link

\text{mod 2 reduction of the sum of the Art invariant for knots constituent}
\( L = k_1 \cup k_2 \) \( 2 \)-comp. ord. ori. link

\( a_i(\cdot) \): \( i \)-th coefficient of \( \nabla(z) \)

**generalized Sato-LEVINE invariant:**

\[ a_3(L) = k(L) \left\{ a_2(k_1) + a_2(k_2) \right\} \]

\( \tilde{\beta}(L) \) \( \text{def} \)

**reduced Arf invariant:**

\[ \overline{\text{Arf}}(L) \overset{\text{def}}{=} \text{Arf}(L) - \left\{ \text{Arf}(k_1) + \text{Arf}(k_2) \right\} \]

proper \( ( \in \{ 0, 1 \} ) \)

**Remark.**

If \( k(L) = 0 \), then

\[
\begin{cases}
    \tilde{\beta}(L) = \beta(L) & \text{Sato-LEVINE inv.} \\
    \overline{\text{Arf}}(L) \equiv \beta(L) \pmod{2}
\end{cases}
\]
§ 2. $C_\theta$-moves on spatial graphs

$C_\theta$-move [Habiro]:

$t_\theta = 1:
\begin{array}{c}
- \quad X \quad C_1 \quad X
\end{array}$

$t_\theta \in \mathbb{Z}_2:

\textbf{This move is closely related to the Vassiliev invariant for knots.}$
$C_2$-move = delta move.
Lemma 2.1.

A $C_\text{K}$-move is realized by self delta moves and ambient isotopies if at least three of the $(k+1)$ strings in it belong to the same spatial edge.
Definition 2.2.
A Ck-move is an adjacent Ck-move.

\[ \text{def} \quad \iff \text{All (k+1) strings in the move belong to exactly (k+1) mutually adjacent spatial edges.} \]

(Here, we regard a loop as two mutually adjacent edges.)
Lemma 2.3.
An adjacent $C_k$-move is realized by $C_k+1$-moves and ambient isotopies.

Remark.

$k=1$ : [Motohashi - Taniyama]

$k=2$ : [Taniyama - Yasuhara]

Lemma 2.3 is a generalization of the facts above.
§ 3. Proof of Main Theorem

(Proof.)

(1) $\Rightarrow$ (2):
It is clear by $(\text{DEH}) \Rightarrow (\#\text{EH}).$

(2) $\Rightarrow$ (3):

Note that:
For $f, g : G \to S^3$ sp. emb.
$f \#\text{EH} g \Rightarrow f|_H \#\text{EH} g|_H$
for $\forall H \subseteq G.$

So we have that (2) $\Rightarrow$ (3).
(3) \implies (4):

Assume that \( G \not\cong \emptyset \) or 00

**Claim 1.** \( G \) is **non**-planar.

![Graphs K5 and K3,3](image)

**Claim 2.** \( G \) is a generalized bouquet.

\( G \): planar graph \( \not\exists \) disjoint cycles

(☺) the assumption & Claim 1.

\[ \iff \quad G \text{ is a} \]

- generalized bouquet
- multipole wheel
- double trident

**Claim 3.** \( G \) is a bouquet.

**Clear.**
(4) \implies (1):

\[
\begin{align*}
(f & : B_n \to S^3 \text{ sp. emb.} \\
(h & : B_n \to S^3 \text{ trivial sp. emb.})
\end{align*}
\]

**Step 1.** $f \sim h$ (clear)

We can regard each of these $C_1$-moves as an adjacent $C_1$-move.

\[
\implies f \sim C_2 h \quad (\text{Lemma 2.3})
\]

**Step 2.** $f \sim C_2 h$

For each of these $C_2$-moves,

- If all strings belong to the same knot, by **Lemma 2.1** we can realize them by self delta moves.

- Otherwise, each of

  We can regard them as an adjacent $C_2$-move

  **Lemma 2.3**

\[
\implies f \sim C_3 h
\]
Step 3. \[ f \rightsquigarrow h \]

For each of these \( C_3 \)-moves, 

By following procedure above repeatedly

Step 2m. \[ f \rightsquigarrow h \]

For each of these \( C_{2m} \)-moves,

\[ f \in f(B_{2m}) \] s.t.

at least three of the \((2m+1)\) strings in the \( C_{2m} \)-move belong to it.

Lemma 2-1

\[ \Rightarrow \]

realized by self delta moves.

\[ f \rightsquigarrow h \]

This completes the proof. \( /// \)