New Hirzebruch-type Invariants from Iterated p-Covers

Jae Choon Cha

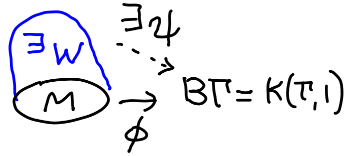
The 7th KAIST Geometric Topology Fair

July 9, 2007

A Hirzebruch-type invariant

Given $\left\{\begin{array}{l} M^3 \text{ closed} \\ \phi\colon \pi_1 M \to \Gamma \\ \mathbb{Z}\Gamma \to \mathcal{K} = \text{(skew) field, char} = 0 \end{array}\right\}$, suppose $(M,\phi) = 0$ in $\Omega_3^{top}(B\Gamma)$

i.e.,



Define $\lambda(M,\phi) = [\lambda_W^{\mathcal{K}}] - [\lambda_W^{\mathbb{Q}}] \in L^0(\mathcal{K})$ = Witt group over $\mathcal{K}$

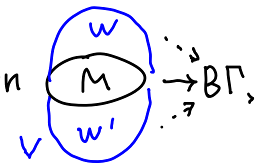
$[\lambda_W^{\mathcal{K}}]$ = $\mathcal{K}$-coefficient intersection form, $[\lambda_W^{\mathbb{Q}}]$ = ordinary intersection form

c.f. Signature defects due to Hirzebruch, Atiyah, Patodi, Singer, ⋯

Proposition: $\lambda(M,\phi)$ is indep. of W if $H_4(\Gamma) = 0$

Well-definedness (independence of W)

Given



need to check: $[\lambda_W^{\mathcal{K}}] - [\lambda_W^{\mathbb{Q}}] = [\lambda_{W'}^{\mathcal{K}}] - [\lambda_{W'}^{\mathbb{Q}}]$

i.e. for $V = W \cup_\partial -W'$, $[\lambda_V^{\mathcal{K}}] = [\lambda_V^{\mathbb{Q}}]$?

An Atiyah-type Lemma: $\left\{\begin{array}{l} V \text{ closed} \\ H_4(\Gamma) = 0 \end{array}\right\} \Rightarrow [\lambda_V^{\mathcal{K}}] = [\lambda_V^{\mathbb{Q}}]$

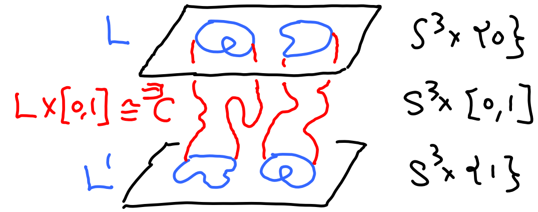
c.f. Signature theorems of Atiyah, Singer, Patodi, ⋯ from "index theory": (twisted signature) = (signature)

Our main example: $\Gamma = \mathbb{Z}_d$, $\mathbb{Z}\Gamma \to \mathcal{K} = \mathbb{Q}(\zeta_d)$, $\zeta_d = \exp\left(\frac{2\pi\sqrt{-1}}{d}\right)$

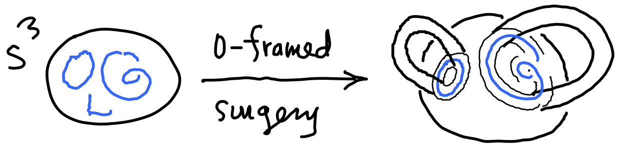
$\rightsquigarrow L^0(\mathbb{Q}(\zeta_d))$ is <u>NOT</u> torsion free!

Link Concordance and Homology cobordism

$L, L' \subseteq S^3$ are **concordant** $\overset{\text{def}}{\Longleftrightarrow}$



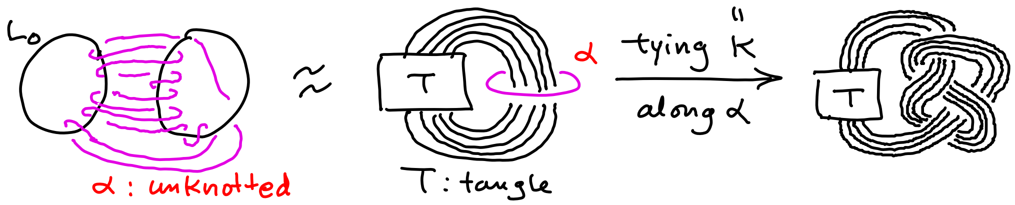
L is **slice** $\overset{\text{def}}{\Longleftrightarrow} L \overset{\text{conc.}}{\sim}$ unlink



$M_L :=$ "**surgery manifold**" of L

Fact: L, L' are concordant $\Rightarrow$ $M_L, M_{L'}$ are homology cobordant

Infection using a knot



Question : How can we detect the effect of infection along $\alpha \in \pi_1(S^3 - L_0)^{(n)} - \pi_1(S^3 - L_0)^{(n+1)}$ for higher n ?

Remarkable results on concordance have been proved by :

Casson, Gordon, Gilmer, B Jiang, Livingston, $\cdots$ ($n=1$: CG-invariant)

Cochran, Orr, Teichner, Harvey, T Kim, Leidy, $\cdots$ ($n>1$: L^2-signature)

Any new technique beyond L^2-signatures ?

e.g. K is "torsion" $\Rightarrow$ no known signatures detect infection for $n>1$

Invariants from p-towers (p : prime)

$M_n \to \cdots \to M_1 \to M_0 = M$ tower of abelian p-covers

$\phi : \pi_1(M_n) \to \mathbb{Z}_d \quad (d = p^a)$ character

Given a p-structure $\mathcal{T} = (\{M_i\}, \phi)$, $\lambda(M_n, \phi)$ is defined.

Remark : (1) For any (nonabelian) p-cover $\tilde{M}$ of M, $\exists \{M_i\}$ s.t. $M_n = \tilde{M}$.

(2) M_n can be an irregular cover of M.

Alternative description : p-virtual character of $\pi_1(M)$

$\phi : H \to \mathbb{Z}_d$, $[G;H] = p^a \Rightarrow \exists$ p-tower $\{M_i\}$

$H \subset G = \pi_1(M)$ s.t. $\pi_1(M_n) = H \subseteq \pi_1(M)$

$\Rightarrow \lambda(M_n, \phi) \in L^0(\mathbb{Q}(\zeta_d))$

Theorem Suppose M and M' are homology cobordant. Then :

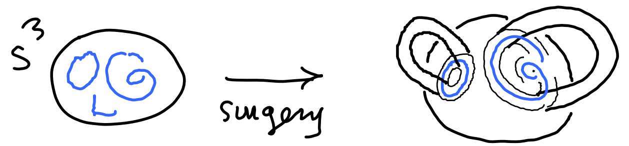
(1) $\{\text{p-structures for } M\} \overset{1-1}{\approx} \{\text{p-structures for } M'\}$

$(\{M_i\}, \phi) \longleftrightarrow (\{M_i'\}, \phi')$

(2) $(M_n, \phi) = 0 \Leftrightarrow (M_n', \phi') = 0$ in $\Omega_3^{top}(B\Gamma)$

If it is the case, $\lambda(M_n, \phi) = \lambda(M_n', \phi')$ in $L^0(\mathbb{Q}(\zeta_d))$

Corollary M = surgery mfd of a slice link $L \Rightarrow \lambda(M_n, \phi) = 0$



M_L = "Surgery manifold" of L

Advantages of our invariants

(1) It extracts geometric information from $\pi_1(M)^{(n)}$ for higher n.

(2) It detects "torsion", as well as elts of ∞ order elts.

$$L^0(\mathbb{Q}(\zeta_d)) \xrightarrow{\text{signature}} L^0(\mathbb{C}) \cong \mathbb{Z}$$

$$L^0(\mathbb{Q}(\zeta_d)) \xrightarrow{\text{discriminant}} \frac{\mathbb{Q}(\zeta_d + \zeta_d^{-1})^\times}{\{z\bar{z} \mid z \in \mathbb{Q}(\zeta_d)^\times\}} \xrightarrow[\text{norm residue symbol}]{(-, D)_q} \mathbb{Z}_2$$

(Artin reciprocity)

$\operatorname{dis} \lambda \underset{\text{def}}{=} (-1)^{\frac{r(r+1)}{2}} \det(\lambda)$

(3) In many interesting cases, $\lambda(M_n, \phi)$ can be computed via a combinatorial algorithm on graphs.

"Exotic" homology cobordism classes of rational 3-spheres

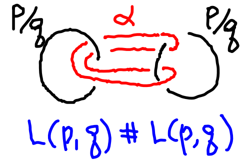
Theorem: $\exists$ rational homology 3-spheres $M_0, M_1, M_2, \cdots$ with the following properties:

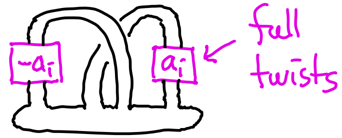
(0) $\exists$ homology equivalence $M_i \to M_0$ for all i.

(1) Known homology cobordism invariants vanish for M_i:
- Wall multisignatures [Gilmer-Livingston]
- Atiyah-Patodi-Singer η-invariants [Levine]
- Cheeger-Gromov L^2-signatures [Harvey]

(2) For $i \neq j$, M_i and M_j are not homology cobordant.

Choose p/q s.t. $L(p,q)$ bounds a rational 4-ball, and let

M_i = 

infected by K_i = 

(along α)

(2-torsion)

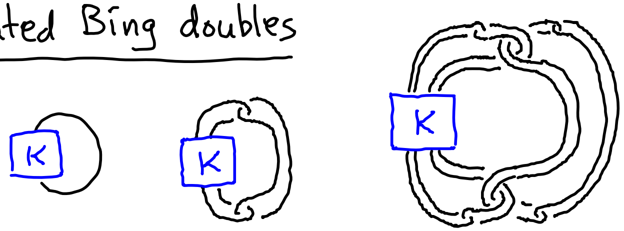
$\left\{ \begin{array}{l} L(p,q) \text{ has vanishing signatures} \\ K_i \text{ is torsion (signature} = 0) \end{array} \right\} \Rightarrow$ Signatures vanish for M_i

Using algebraic number theory, we construct $\{a_i\}$ together with "dual primes" $\{q_i\}$ s.t.

$$\left(\begin{array}{c} \text{norm residue symbol of} \\ \text{discriminant of } M_i \text{ w.r.t. } q_j \end{array} \right) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$\Rightarrow \{M_i\}$ realizes $\mathbb{Z}_2^\infty \subseteq L^0(\mathbb{Q}(\zeta_d))$

Iterated Bing doubles

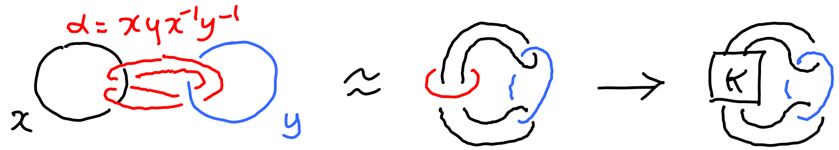


obtained by infection along $\alpha \in G^{(n)}$

$K \to BD(K) \longrightarrow BD_2(K) \to \cdots \to BD_n(K)$

Question: When is $BD_n(K)$ slice? (Many known invariants vanish!)

Difficulty: the complication is invisible on "abelianization"

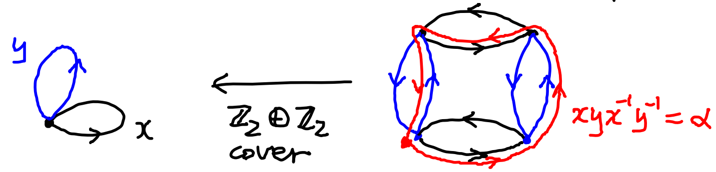


$x \cup y$ = trivial link

$BD(K)$

But, the infection curve α survives in H_1(iterated p-covers):

e.g. $n=1$:



This enables us to detect non-slice iterated Bing doubles using our results:

Theorem If the Levine-Tristram signature $\sigma_K : S^1 \to \mathbb{Z}$ is nontrivial, then $BD_n(K)$ is not slice for any n.

($\Rightarrow$ [Harvey, Teichner] If $\int_{S^1} \sigma_K \neq 0$, then $BD_n(K)$ is not slice
(The proof uses L^2-signatures))

2-torsion

L is called **2-torsion** if $L \# L$ is slice ($\Rightarrow L$ is conc. to $-L$)

e.g. $K = 4_1 =$  $\Rightarrow K \approx -K$ (amphichiral) $\Rightarrow K$ is 2-torsion

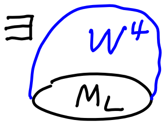
Facts: (1) K 2-torsion $\Rightarrow BD_n(K)$ 2-torsion

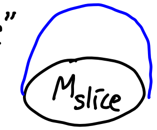
(2) Known (signature) invariants fail to distinguish $BD_n(K)$ from a slice link when K is torsion

Question [Teichner, Schneidermann, Cimasoni, Friedl, Cochran, …]
Is $BD(4_1)$ a slice link?

Theorem There are infinitely many amphichiral K (including 4_1) s.t. $BD_n(K)$ is not slice for all n.

Cochran-Orr-Teichner's solvable filtration

L is **(n)-solvable** $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ 

whose $\mathbb{Z}\left[\frac{\pi_1 W}{\pi_1 W^{(n)}}\right]$-coeff. duality "looks like" 

slice disk exterior

COT filtration: $\underbrace{\{(n)\text{-solv. links}\}/\text{conc.}}_{= \mathcal{F}_{(n)}}$

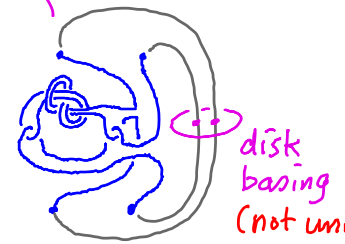
$$\{0\} \subseteq \cdots \subseteq \mathcal{F}_{(n.5)} \subseteq \mathcal{F}_{(n)} \subseteq \cdots \subseteq \mathcal{F}_{(0.5)} \subseteq \mathcal{F}_{(0)} \subseteq \{\text{links}\}/\text{conc.}$$

Question: Is there any (nontrivial) torsion in higher terms?

Theorem: $L \in \mathcal{F}_{(n.5)} \Rightarrow$ for any p-structure $(\{M_i\}, \phi)$ of height n, $\lambda(M_n, \phi) = 0$.

Theorem: $\exists$ infinitely many 2-torsion $L \in \mathcal{F}_{(n)} - \mathcal{F}_{(n+1.5)}$

String link concordance "group"

$\hat{\beta} =$ 

$$\mathcal{C}^{SL} := \frac{\{\text{string links}\}}{\text{concordance}}$$

product: concatenation

inverse: mirror image

c.f. $\mathcal{C}^L = \dfrac{\{\text{links}\}}{\text{Concordance}}$

product: connected sum

… not well defined!

Def β is an **$\hat{F}$-string link** if μ induces $\widehat{\pi_1(X)} \cong \widehat{\pi_1(S^3 - \hat{\beta})}$

where $\hat{G}$ = "algebraic closure of G", and longitudes $= 0$ in $\widehat{\pi_1(X)}$
[Levine, Vogel, C, …]

$\hat{\mathcal{C}}^{SL} :=$ subgp gen. by $\hat{F}$-string links $\subseteq \mathcal{C}^{SL}$

COT filtration: $\{0\} \subseteq \cdots \subseteq \hat{\mathcal{F}}^{SL}_{(n.5)} \subseteq \underbrace{\hat{\mathcal{F}}^{SL}_{(n)}}_{\parallel} \subseteq \cdots \subseteq \hat{\mathcal{F}}^{SL}_{(0.5)} \subseteq \hat{\mathcal{F}}^{SL}_{(0)} \subseteq \hat{\mathcal{C}}^{SL}$

subgp gen. by (n)-solvable $\hat{F}$-string links

Proposition: For $\beta \in \hat{\mathcal{C}}^{SL}$, μ induces a bijection

$$\left\{\begin{array}{c}\text{p-structures } (\{X_i\}, \theta) \\ \text{for } X\end{array}\right\} \approx \left\{\begin{array}{c}\text{p-structures } (\{M_i\}, \phi) \\ \text{for } M_{\hat{\beta}}\end{array}\right\}$$

For $\mathcal{J} = (\{X_i\}, \theta)$ of height n, define $\lambda_{\mathcal{J}}(\beta) := \lambda(M_n, \phi)$.

Theorem: For any $\mathcal{J}=(\{x_i\},\theta)$ with height of $\{x_i\}\le n$, $\lambda_{\mathcal{J}}(\beta)$ induces a group homomorphism

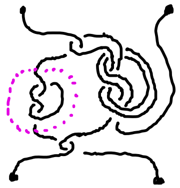
$$\lambda_{\mathcal{J}}: \widehat{\mathcal{F}}^{SL}_{(n)}/\widehat{\mathcal{F}}^{SL}_{(n.5)} \subseteq \widehat{\mathcal{C}}^{SL}/\widehat{\mathcal{F}}^{SL}_{(n.5)} \longrightarrow L^0(\mathbb{Q}(\mathcal{J}_d))$$

We say $\mathcal{J}$ is locally trivial if θ kills lifts of powers of x_i = ith circle of $X=\bigvee^m S^1$.

Theorem: If $\mathcal{J}$ is locally trivial,

$\lambda_{\mathcal{J}}(\beta)=\lambda_{\mathcal{J}}(\beta$ with "local knots")

local knot



Remark $\{\text{links}\}/\text{local knots}$ $\rightsquigarrow$ sophistication peculiar to "links"

Theorem: abelianization of $\dfrac{\widehat{\mathcal{F}}^{SL}_{(n)}}{\widehat{\mathcal{F}}^{SL}_{(n.5)}\cdot\langle\text{local knots}\rangle} \supseteq \mathbb{Z}^\infty$

{n-solv. boundary links} $\parallel$

$\widehat{\mathcal{F}}^{SL}_{(n)}/\widehat{\mathcal{F}}^{SL}_{(n.5)}$ $\cup\!\!|$

C.f. Harvey defined $\rho_n: \mathcal{BF}^{SL}_{(n)}/\mathcal{BF}^{SL}_{(n.5)} \to \mathbb{R}$

and using it, showed (abelianization of $\mathcal{BF}^{SL}_{(n)}/\mathcal{BF}^{SL}_{(n.5)}$) $\supseteq \mathbb{Z}^\infty$

Theorem: The kernel of ρ_n is large: abelianization of $\operatorname{Ker}\rho_n \supseteq \mathbb{Z}^\infty$ (modulo local knots)

i.e. $\exists$ infinitely many "independent" string links in $\operatorname{Ker}\rho_n$.

Independence of links

Theorem: $\exists$ infinitely many $L_i\in\mathcal{F}_{(n)}$ "independent mod $\mathcal{F}_{(n.5)}$" w.r.t. connected sum in the following sense:

$\#_i\, a_i L_i \in \mathcal{F}_{(n.5)}$ for some disk basings $\Rightarrow a_i=0\ \forall i$.

Theorem: $\exists$ infinitely many 2-torsion $L_i\in\mathcal{F}_{(n)}$ which are "independent mod $\mathcal{F}_{(n+1.5)}$" in the following sense:

For $a_i\in\{0,1\}$, $\#_i\, a_i L_i \in \mathcal{F}_{(n+1.5)}$ for some disk basings

$\Rightarrow a_i=0\ \forall i$.

Thank You!