New Hirzebruch-type Invariants from Iterated p-Covers

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A Hirzebruch-type invariant

Given $M^3$ closed
\[ \phi: \pi, M \to \Gamma \]
\[ \mathbb{Z}\Gamma \to k = (\text{skew}) \text{ field} \]
\[ \text{char } = 0 \]
\[ \exists \mathcal{W} \rightarrow \mathcal{B} \Gamma = k(\mathbb{Q}, 1) \]

Define $\lambda(M, \phi) = [\lambda^w]\mathcal{W} - [\lambda^w]\mathcal{W} \in L^0(k) = \text{Witt group over } k$

\text{K-coefficient ordinary intersection form}
\text{homology intersection form}

$c.f.$ Signature defects due to Hirzebruch, Atiyah, Patodi, Singer, ...

Proposition: $\lambda(M, \phi)$ is independent of $W$ if $H_4(\Gamma) = 0$

Well-definedness (independence of $W$)

Given $W \rightarrow B \Gamma$, need to check: $[\lambda^w]\mathcal{W} - [\lambda^w]\mathcal{W}$
\[ \text{i.e. for } V = W U - W' \]
\[ [\lambda^w]\mathcal{V} = [\lambda^w]\mathcal{W} \]

An Atiyah-type Lemma:
\[ \begin{cases} V \text{ closed} \\ H_4(\Gamma) = 0 \end{cases} \Rightarrow [\lambda^w]\mathcal{W} = [\lambda^w]\mathcal{W} \]

$c.f.$ Signature theorems of Atiyah, Singer, Patodi, ...
from "index theory": (twisted signature) = (signature)

Our main example: $\Gamma = \mathbb{Z}_2$, $Z\Gamma \to \mathbb{Q}(5_2)$, $5_2 = \exp(\frac{\pi i}{\sqrt{2}})$
\[ \Rightarrow L^0(\mathbb{Q}(5_2)) \text{ is NOT torsion free!} \]

Link Concordance and Homology cobordism

$L, L' \subseteq S^3$ are concordant
$L \times [0, 1] \cong \mathbb{C}$
$L \cong \text{slice}$
$L \sim \text{unlink}$

$L \times [0, 1]$ is disjoint from $S^3$ for $0$-framed surgery

$S^3 \stackrel{\text{O}(L)}{\longrightarrow} \text{Lsurgeries}$

$M_L := \text{"surgery manifold" of } L$

Fact: $L, L'$ are concordant $\Rightarrow M_L, M_L'$ are homology cobordant

$\exists W \rightarrow \mathcal{B} \Gamma = k(\mathbb{Q}, 1)$
**Invariants from p-towers** \( (p: \text{prime}) \)

\[ M_n \xrightarrow{\ldots} M_1 \xrightarrow{\mu} M_0 = M \quad \text{tower of abelian p-covers} \]

\( \phi: \pi_1(M_n) \to \mathbb{Z}_d \quad (d=p^a) \quad \text{character} \)

*Given a p-structure \( \mathcal{J} = (\gamma; M, \phi) \), \( \lambda(M_n, \phi) \) is defined.*

**Remark:** (1) For any (nonabelian) p-cover \( \widetilde{M} \) of \( M \), \( \exists \gamma; M, \phi \) s.t. \( M \simeq \widetilde{M} \).

(2) \( M_n \) can be an irregular cover of \( M \).

**Alternative description:** \( p \)-virtual character of \( \pi_1(M) \)

\[ \phi: H \to \mathbb{Z}_d, \quad [G; H] = p^a \quad \Rightarrow \quad \exists \ p\text{-tower} \quad \gamma; M \]

\[ G = \pi_1(M), \quad \text{s.t.} \quad \pi_1(M_n) = H \subseteq \pi_1(M) \]

\[ \Rightarrow \quad \lambda(M_n, \phi) \in L^0(\mathbb{Q}(\sqrt{d})) \]

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**Advantages of our invariants**

(1) It extracts geometric information from \( \pi_1(M)^{(n)} \) for higher \( n \).

(2) It detects "torsion", as well as elements of order \( d \) of \( \mathbb{Z} \).

\[ \text{Signature} \quad L^0(C) \cong \mathbb{Z} \]

\[ L^0(\mathbb{Q}(\sqrt{d})) \]

\[ \Delta \quad \text{discriminant} \quad \mathbb{Q}(\sqrt{5d + 5d^{-1}}) \]

\[ \left\{ \begin{array}{l}
\text{Norm residue symbol} \\
\text{(Artin reciprocity)}
\end{array} \right\} \]

\[ \text{Norm} \quad \mathbb{Z}_2 \]

(3) In many interesting cases, \( \lambda(M_n, \phi) \) can be computed via a combinatorial algorithm on graphs.
"Exotic" homology cobordism classes of rational 3-spheres

**Theorem**: \( \exists \) rational homology 3-spheres \( M_0, M_1, M_2, \ldots \)
with the following properties:

1. Homology equivalence \( M_i \to M_0 \) for all \( i \).
2. Known homology cobordism invariants vanish for \( M_i \):
   - Wall multisymp signedatures \([\text{Gilmer-Livingston}]\)
   - Atiyah-Patodi-Singer \( \eta \)-invariants \([\text{Levine}]\)
   - Cheeger-Gromov \( L^2 \)-signatures \([\text{Harvey}]\)
3. For \( i \neq j \), \( M_i \) and \( M_j \) are not homology cobordant.

Choose \( P/\mathbb{Q} \) s.t. \( L(P,\mathbb{Q}) \) bounds a rational 4-ball, and let

\[
M_i = L(P,\mathbb{Q}) \# L(P,\mathbb{Q}) \quad \text{infected by } K_i = -2 \cdot \text{full twist}
\]

\( L(P,\mathbb{Q}) \) has vanishing signatures \rightarrow \text{Signatures vanish for } M_i

\( K_i \) is torsion (signature = 0)

Using algebraic number theory, we construct \( \{a_i\} \) together with "dual primes" \( \{\beta_i\} \) s.t.

\[
\left( \text{norm residue symbol of } \beta_i \right) = 1 \quad \text{if } i \neq j \quad \text{and} \quad 0 \quad \text{if } i = j
\]

\( \{M_i\} \) realizes \( \mathbb{Z}_2^\infty \subseteq L^0(\mathbb{Q}(S_4)) \)

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**Iterated Bing doubles**

\[
K \to BD(K) \to BP_2(K) \to \cdots \to BD_n(K)
\]

Question: When is \( BD_n(K) \) slice? (Many known invariants vanish!)

**Difficulty**: the complication is invisible on "abelianization"

\[
x = xyx^{-1}y^{-1}
\]

\( x^2 y = \text{trivial link} \)

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But, the infection curve \( \alpha \) survives in \( H_1(\text{iterated p-covers}) \):

\[\text{e.g., } n=1: y \to x \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \text{ cover} \]

\[xyx^{-1}y^{-1} = \alpha \]

This enables us to detect non-slice iterated Bing doubles using our results:

**Theorem**: If the Levine-Tristram signature \( \sigma_k: S^1 \to \mathbb{Z} \) is nontrivial, then \( BD_n(K) \) is not slice for any \( n \).

\( \Rightarrow [\text{Harvey, Teichner}] \text{ If } \int S^1 \sigma_k \neq 0, \text{ then } BD_n(K) \text{ is not slice} \)

(The proof uses \( L^2 \)-signatures.)
2-torsion

L is called 2-torsion if L # L is slice \( \Rightarrow L \) is conc to \(-L\)

e.g. \( K = 4_i = \begin{array}{c}
1
\end{array} \Rightarrow K \approx -K \Rightarrow K \) is 2-torsion
(amphichiral)

Facts:
1. \( K \) 2-torsion \( \Rightarrow BD_n(K) \) 2-torsion
2. Known (signature) invariants fail to distinguish
   \( BD_n(K) \) from a slice link when \( K \) is torsion

Question [Teichner, Schneiderman, Giacinto, Friedl, Cochran, …]
Is \( BD(4_i) \) a slice link?

Theorem: There are infinitely many amphichiral \( K \) (including \( 4_i \))
   s.l. \( BD_n(K) \) is not slice for all \( n \).

Cochran-Orr-Teichner's solvable filtration

\( L \) is \( (n) \)-solvable \( \iff \exists \mathcal{U}^4 \) whose \( \mathbb{Z}_{[\mathcal{U}^4]} \)-coeff. duality

\( (n) \)-solvable \( \text{def} \) \( \begin{array}{c}
\mathcal{U}^4
\end{array} \) “looks like” \( \begin{array}{c}
\text{slice}
\text{disk}
\text{exterior}
\end{array} \)

\( \mathcal{U}^4 \)

COT filtration:
\( \{ \text{(n)-solv. links} \} / \text{conc.} \)

\( \mathcal{U}^4 \subseteq \cdots \subseteq \mathcal{U}^3 \subseteq \mathcal{U}^2 \subseteq \mathcal{U}^1 \subseteq \{ \text{links} \} / \text{conc.} \)

Question: Is there any (nontrivial) torsion in higher terms?

Theorem: \( L \in \mathcal{F}(n, 5) \) \( \Rightarrow \) for any \( p \)-structure \( (\mathcal{M}, \phi, \theta) \)
   of height \( n \), \( \lambda(M_n, \phi) = 0 \).

Theorem: \( \exists \) infinitely many 2-torsion \( L \in \mathcal{F}(n_0) \cap \mathcal{F}(n_1) \).

String link concordance "group"

- \( \mu \) : meridian map
- \( \beta = \begin{array}{c}
\text{meridian map}
\end{array} \)

\( \mathcal{E}_{SL} := \{ \text{string links} \} / \text{concordance} \)

product: concatenation
inverse: mirror image

\( \mathcal{E}_{SL} := \text{subgp gen. by } \{ \beta \} \subseteq \mathcal{E}_{SL} \)

COT filtration:
\( \mathcal{U}^4 \subseteq \cdots \subseteq \mathcal{U}^3 \subseteq \mathcal{U}^2 \subseteq \mathcal{U}^1 \subseteq \mathcal{U}^0 \subseteq \mathcal{E}_{SL} \)

subgp gen. by \( (n) \)-solvable \( \# \)-string links

Proposition: For \( \beta \in \mathcal{E}_{SL} \), \( \mu \) induces a bijection

\( \{ p \text{-structures } (\mathcal{X}, \phi, \theta) \} \approx \{ p \text{-structures } (\mathcal{M}, \phi, \theta) \} \)

for \( \mathcal{X} = (\mathcal{X}, \phi, \theta) \) of height \( n \), define \( \lambda_{\mathcal{X}}(\beta) := \lambda(M_n, \phi) \).

Def. \( \beta \) is an \( \mathcal{F} \)-string link if \( \mu \) induces \( \overline{\pi}_1(\mathcal{X}) = \overline{\pi}_1(S^3 - \beta) \)

where \( \overline{\pi}_1 = \text{"algebraic closure of } \overline{\pi}_1(\mathcal{X}) \]

[Levine, Vogel, C, …]

\( \hat{\mathcal{E}}_{SL} := \text{subgp gen. by } \mathcal{F} \)-string links \( \subseteq \mathcal{E}_{SL} \)

Subgroup generated by \( (n) \)-solvable \( \# \)-string links

Proposition: For \( \beta \in \hat{\mathcal{E}}_{SL} \), \( \mu \) induces a bijection

\( \{ p \text{-structures } (\mathcal{X}, \phi, \theta) \} \approx \{ p \text{-structures } (\mathcal{M}, \phi, \theta) \} \)

For \( \mathcal{X} = (\mathcal{X}, \phi, \theta) \) of height \( n \), define \( \lambda_{\mathcal{X}}(\beta) := \lambda(M_n, \phi) \).
Theorem: For any $\mathcal{I} = (x_1, \ldots, x_n)$ with height of $\{x_i\} \leq n$, $\lambda_\mathcal{I}(\beta)$ induces a group homomorphism

$$\lambda_\mathcal{I} : \frac{\widehat{F}_n}{\widehat{\mathcal{F}}_{(n,5)}} \to \mathcal{L}(S^3)$$

We say $\mathcal{I}$ is locally trivial if $\Theta$ kills lifts of powers of $x_i = i^{th}$ circle of $X = \sqrt{m \bar{S}^1}$. 

Theorem: If $\mathcal{I}$ is locally trivial, $\lambda_\mathcal{I}(\beta) = \lambda_\mathcal{I}(\beta$ with "local knots")

Remark: $\mathcal{I}$/local knots $\leadsto$ sophistication peculiar to "links"

Theorem: abelianization of $\frac{\widehat{B}_n}{\widehat{B}_{(n,5)}} \cdot \langle \text{local knots} \rangle \cong \mathbb{Z}^\infty$

c.f. Harvey defined $p_n : \frac{\partial B_n}{\partial B_{(n,5)}} \to \mathbb{R}$ and using it, showed (abelianization of $\frac{\partial B_n}{\partial B_{(n,5)}}$) $\cong \mathbb{Z}^\infty$

Theorem: The kernel of $p_n$ is large: abelianization of $\text{Ker} p_n \cong \mathbb{Z}^\infty$

i.e. $\exists$ infinitely many "independent" string links in $\text{Ker} p_n$. 

Independence of links

Theorem: $\exists$ infinitely many $L_i \in \mathcal{F}_{(n,5)}$ "independent mod $\mathcal{F}_{(n,5)}$" w.r.t. connected sum in the following sense:

$\# a_i L_i \in \mathcal{F}_{(n,5)}$ for some disk basings $\implies a_i = 0 \forall_i$.

Theorem: $\exists$ infinitely many 2-torsion $L_i \in \mathcal{F}_{(n,5)}$ which are "independent mod $\mathcal{F}_{(n+1,5)}$" in the following sense:

For $a_i \in \{0, 1\}$, $\# a_i L_i \in \mathcal{F}_{(n+1,5)}$ for some disk basings

$\implies a_i = 0 \forall_i$. 

Thank You!