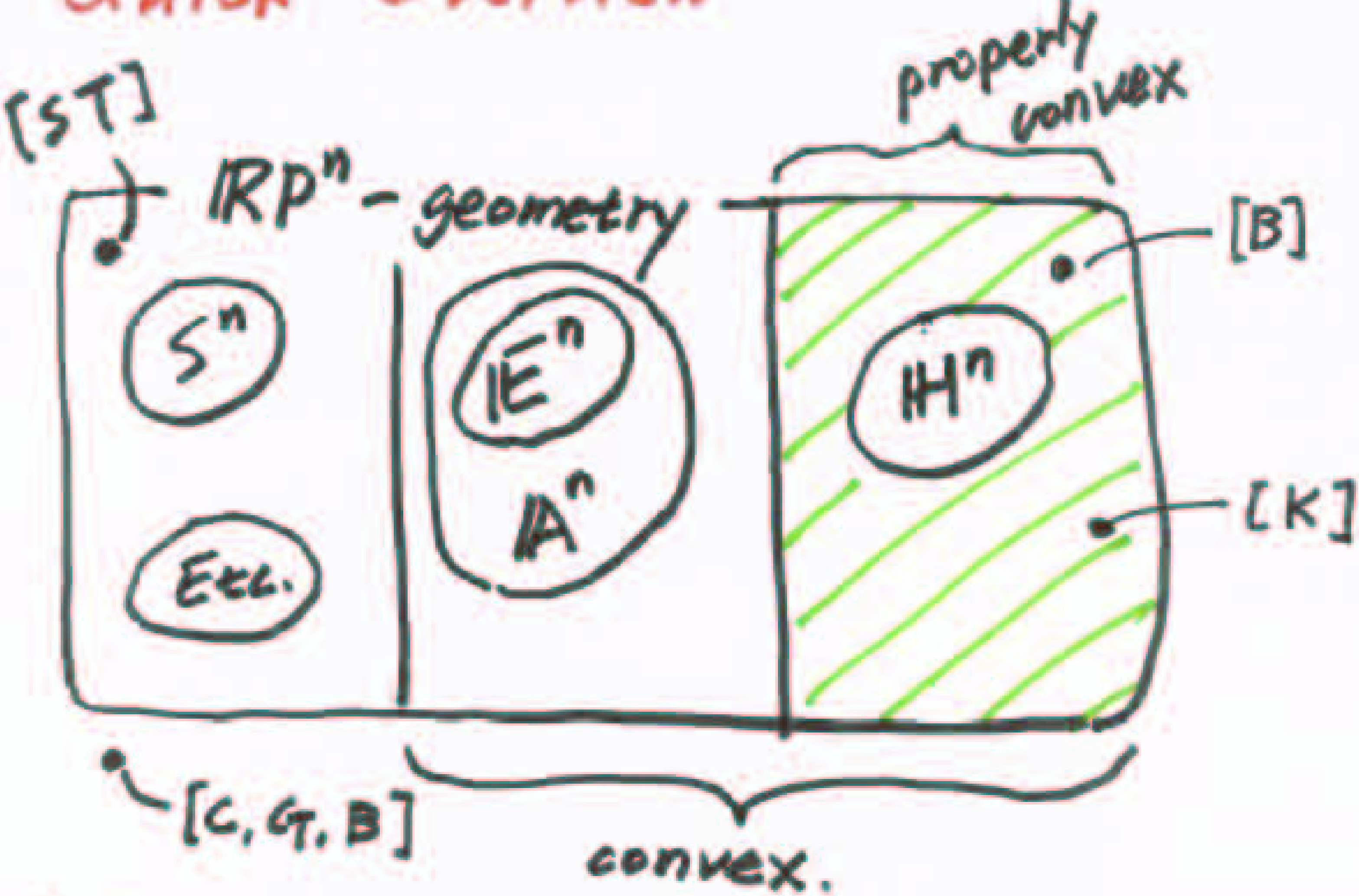
Fundamental Domains for Properly Convex Real Projective Structures

Jaejeong Lee

UC Davis

(Advisor: Misha Kapovich)

Quick Overview



[B] Benoist '06, '04

[K] Kapovich '06

[ST] Sullivan-Thurston '83

[C, G, B] Cooper, Goldman, Benoist, '02 (?)

Issue : Find new interesting examples in the shaded area.

How to construct examples?

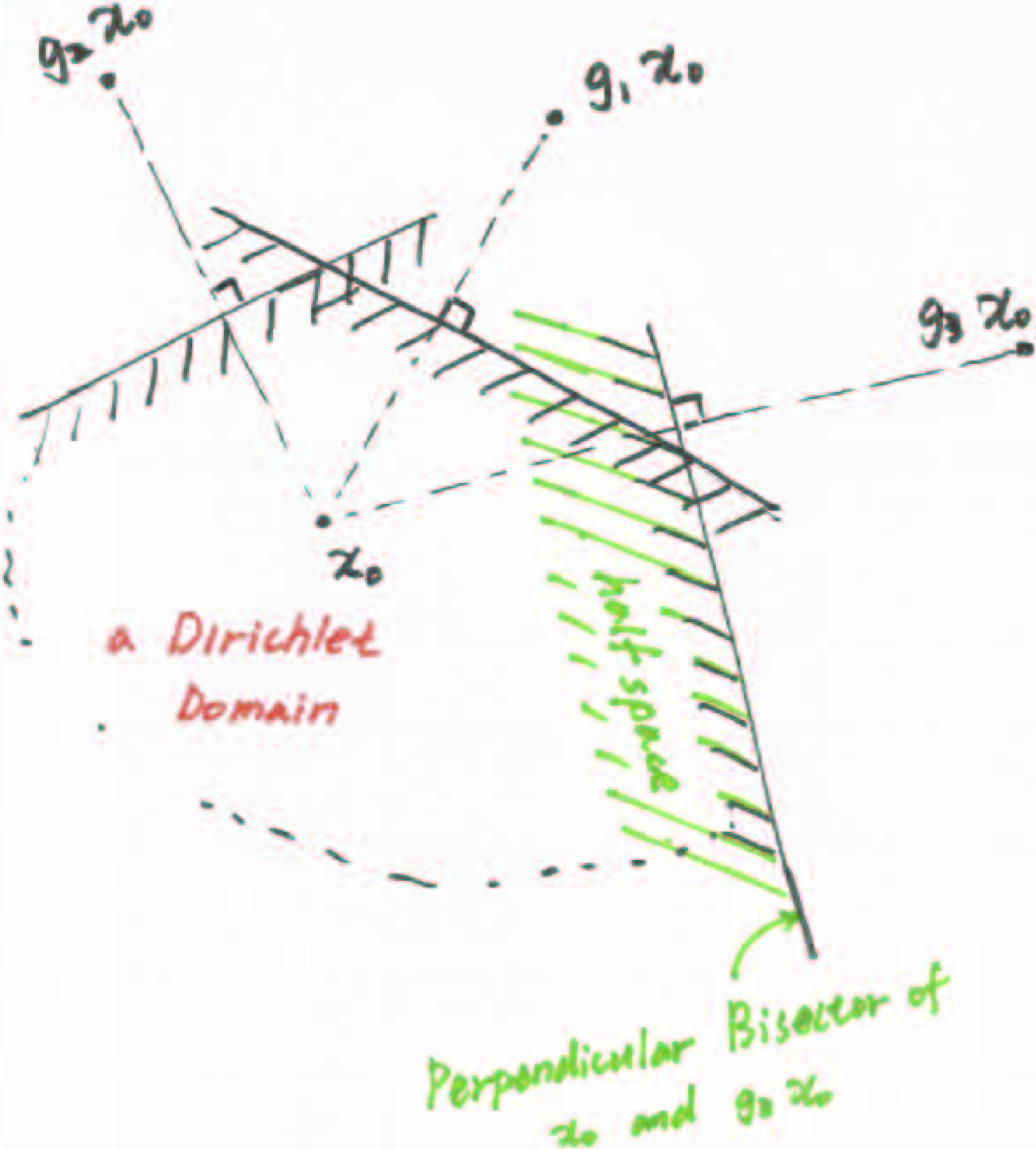
In constant curvature cases
S^n, $\mathbb{E}^n$, $\mathbb{H}^n$,
Poincaré Fundamental Polyhedron Theorem is, in principle, the most general method for constructing examples,

because

these geometries always have convex, locally finite. (hence polyhedral) fundamental domains.

i.e.

Dirichlet Domains.


$g_2 x_0$
$g_1 x_0$
$g_3 x_0$
x_0
a Dirichlet Domain
half space
Perpendicular Bisector of
x_0 and $g_3 x_0$

Real Projective Structure.

- $\mathbb{R}^{n+1}$: real vector space.

- $\mathbb{RP}^n = \{$ 1-dim'l subspaces of $\mathbb{R}^{n+1} \}$

$$= (\mathbb{R}^{n+1} - 0) / \mathbb{R}^*$$

$$\mathrm{Aut}(\mathbb{RP}^n) = GL_{n+1}\mathbb{R} / \mathbb{R}^* := PGL_{n+1}\mathbb{R}$$

- $S^n = \widetilde{\mathbb{RP}}^n = \{$ rays from $0 \}$

$$= (\mathbb{R}^{n+1} - 0) / \mathbb{R}^+$$

$$\mathrm{Aut}(S^n) = \widetilde{\mathrm{Aut}(\mathbb{RP}^n)} = \mathbb{Z}_2 \rtimes \mathrm{Aut}(\mathbb{RP}^n)$$

$$= GL_{n+1}\mathbb{R} / \mathbb{R}^+ := SL^{\pm}_{n+1}\mathbb{R}.$$

- <u>Fact</u> $(\mathbb{RP}^n, \mathrm{Aut}\,\mathbb{RP}^n)$-str.

$$\underset{1-1}{\longleftrightarrow} (S^n, \mathrm{Aut}\, S^n)\text{-str.}$$

(Properly) Convex subsets $\subset \mathbb{S}^n$.

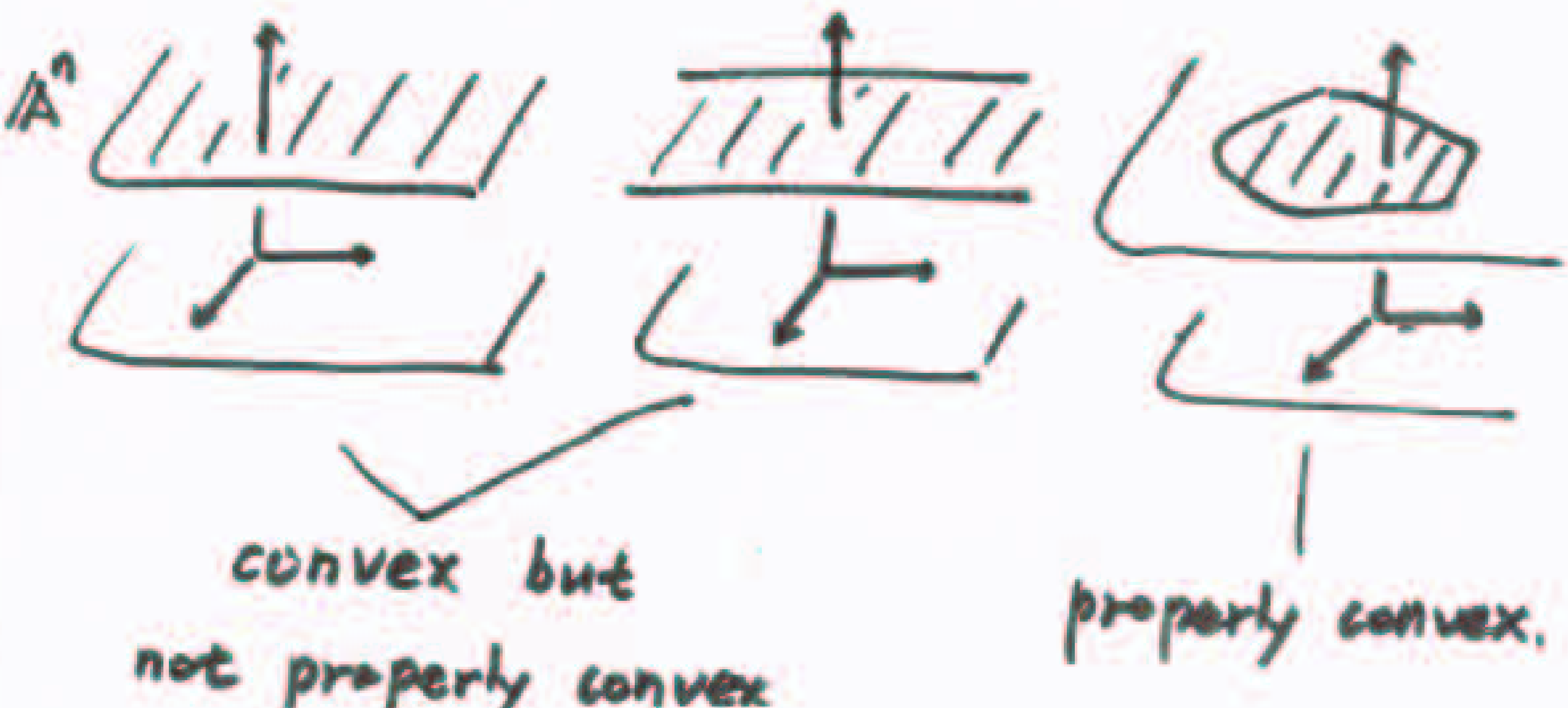
Def A subset $C \subset \mathbb{S}^n$ is

convex if

$$\Lambda_C = p^{-1}(C) \subset \mathbb{R}^{n+1} \text{ is convex,}$$

properly convex if, in addition,

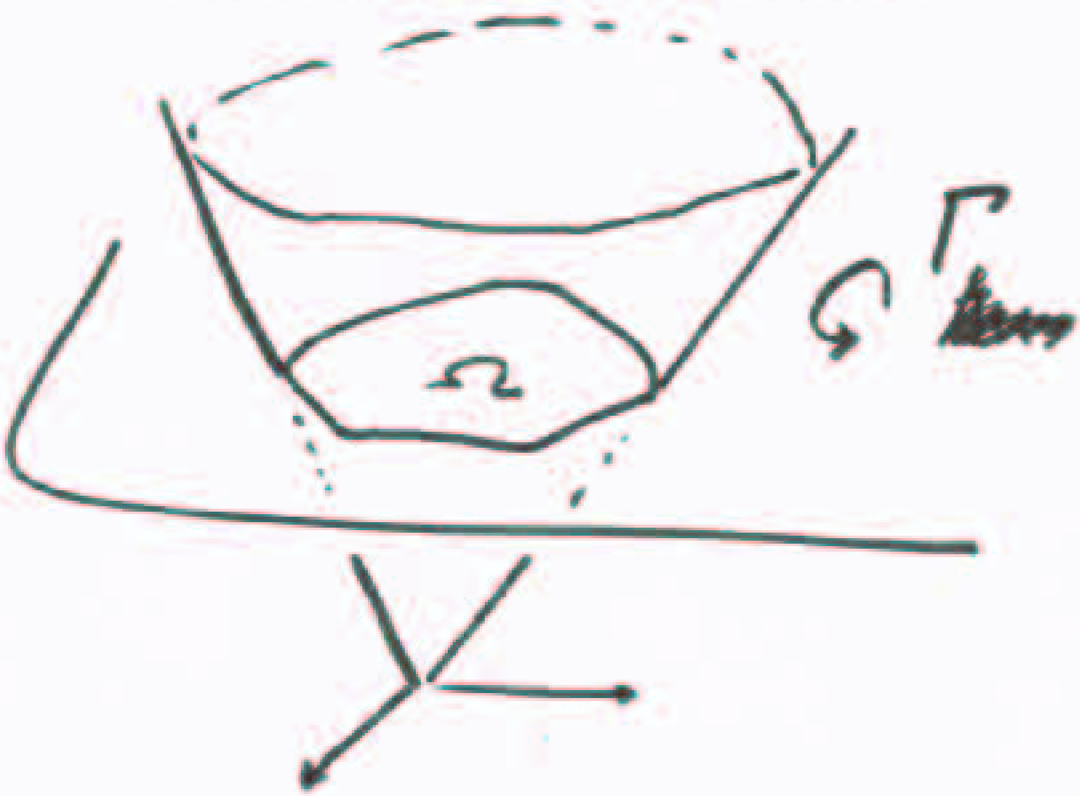
Λ_C contains no complete affine line.



Properly Convex $\mathbb{RP}^n$-manifold

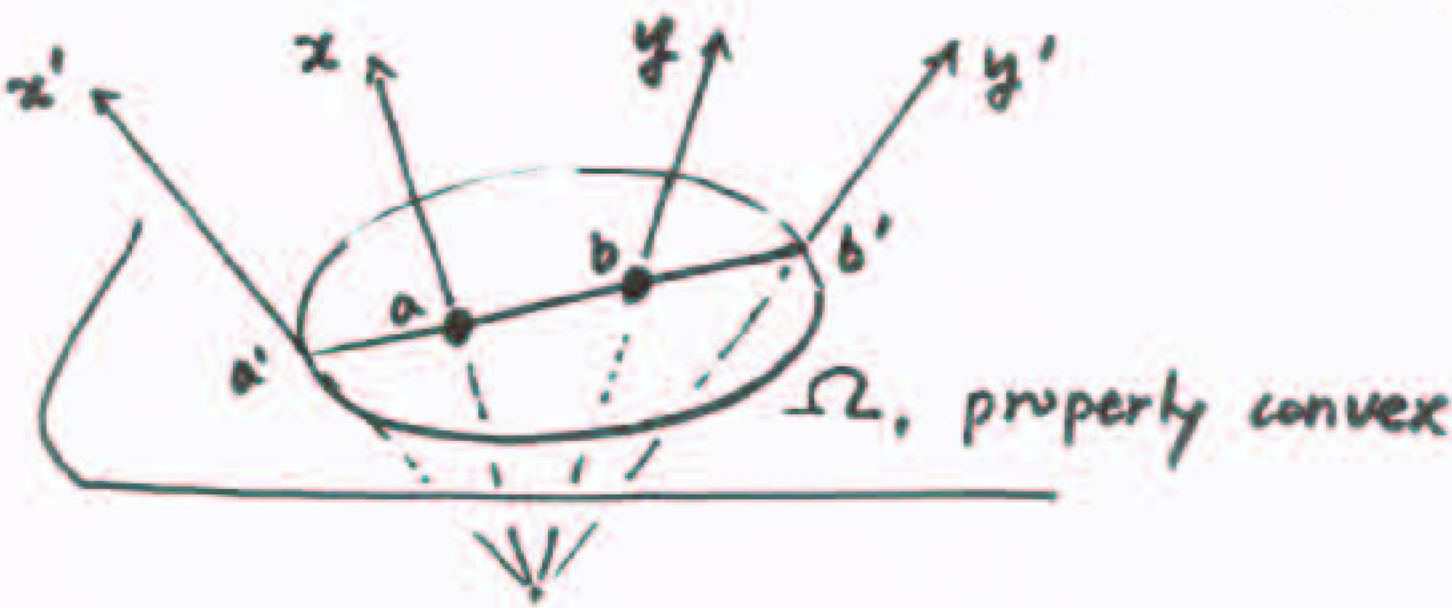
$$M = \Omega/_\Gamma$$

- $\Gamma \curvearrowright \Omega$
- $\Omega \subset \mathbb{S}^n$: properly convex domain
- $\Gamma \subset \mathrm{Aut}(\Omega) \subset \mathrm{Aut}(\mathbb{S}^n)$,
  discrete, torsion-free



Hilbert metric

= natural invariant metric on Ω.



$$d_\Omega(x, y) = \frac{1}{2} \log [x', x, y, y'],$$

where $[x', x, y, y'] = \dfrac{|ab'|}{|aa'|} \cdot \dfrac{|ba'|}{|bb'|}$

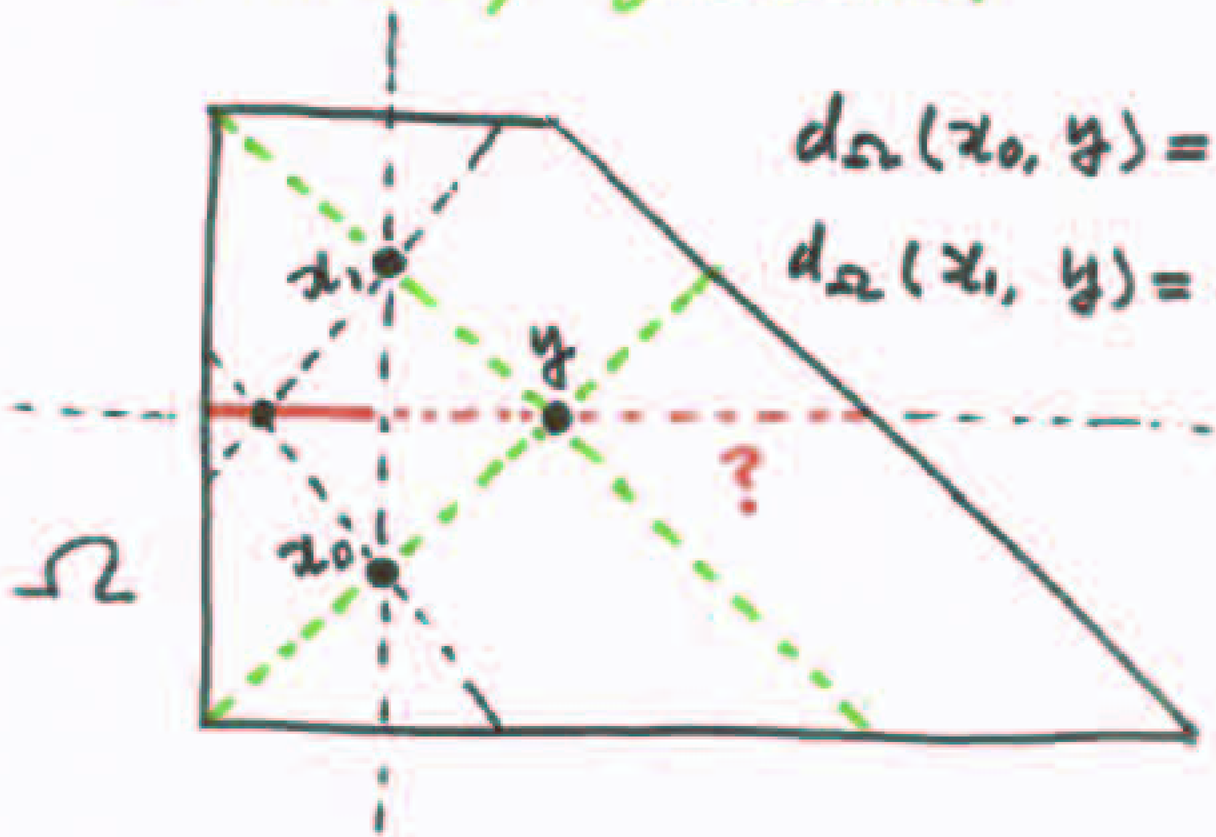
- d_Ω, (genuine) metric. Finsler.
- cross ratio, inv. under $\mathrm{Aut}(S^n)$
  $\Rightarrow \mathrm{Aut}(\Omega) \curvearrowright \Omega$ by isometry.
- proper, complete
- geodesics = lines.

Problem of using Hilbert metric for constructing Dirichlet domain.

The Hilbert metric d_Ω is the best possible we can imagine,

but

one of its disadvantage is that bisectors w.r.t. d_Ω is not flat, i.e., not totally geodesic.



$d_\Omega(x_0, y) = \frac{1}{2} \log 4$

$d_\Omega(x_1, y) = \frac{1}{2} \log 3$

- In fact,
  for a fairly general simply-connected metric space M,
  if all bisectors are totally geodesic,
  then $M = S^n, \mathbb{E}^n, \mathbb{H}^n$.
  (Busemann, '50s (?)).

- Nevertheless,
  there is a way to get around this situation.

- To motivate this construction,
  let's review why bisectors in
  $$\mathbb{H}^n$$
  are totally geodesic.

Hyperboloid model for $\mathbb{H}^n$.

In $\mathbb{R}^{n+1}$, consider

$$L(x, y) = x_1 y_1 + \cdots + x_n y_n - x_{n+1} y_{n+1},$$

Lorentzian inner product

- Lorentzian cone $= \{ x \mid L(x, x) < 0 \}$
- $\mathbb{H}^n$ = upper-sheet of hyperboloid
  $= \{ x \mid L(x, x) = -1,\ x_{n+1} > 0 \}$

- Cauchy – Schwartz:

  for $x, y \in$ Lorentzian cone

  $$L(x, y) \leq L(x, x)^{\frac{1}{2}} L(y, y)^{\frac{1}{2}}$$

- Thus, if $x, y \in \mathbb{H}^n$, then

  $$L(x, y) \leq i \cdot i = -1$$

- We set

  $$-L(x, y) = \cosh d_{\mathbb{H}^n}(x, y).$$

- One checks

  $$d_{\mathbb{H}^n} = d_{Hilbert}$$

  is a Riemannian metric of curv $\equiv -1$.

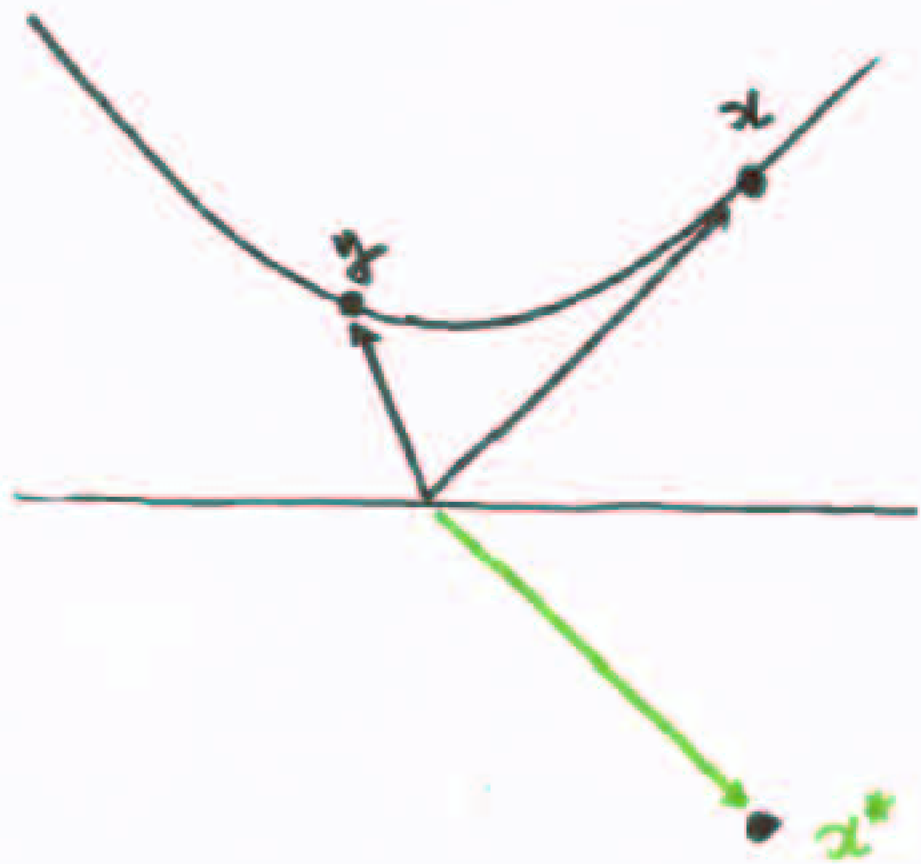
More radical reason for why bisectors in $\mathbb{H}^n$ are tot. geod.

- Set $x^* = Lx$, $L = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$,

then

$$L\langle x, y\rangle = \langle x^*, y\rangle_E.$$

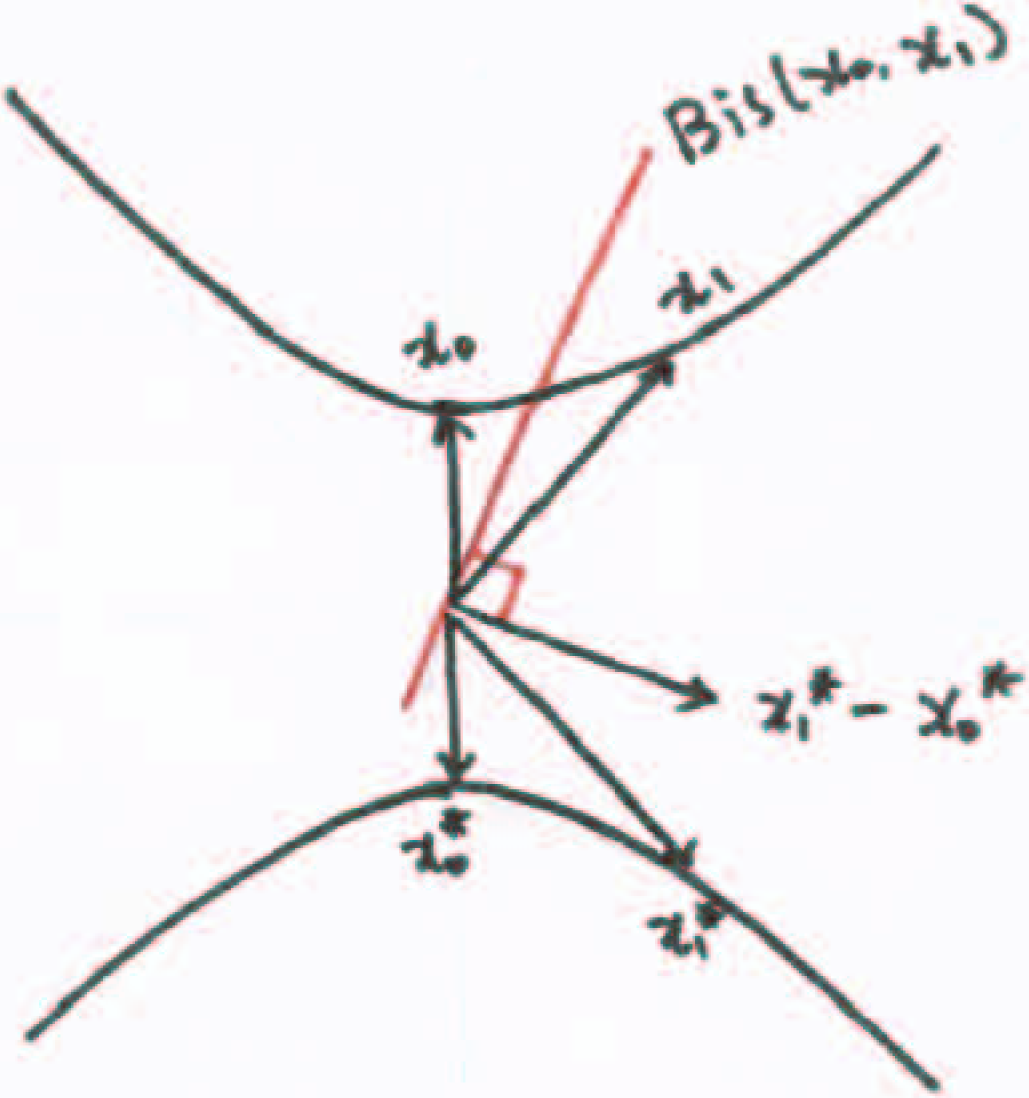
$$= x_1 y_1 + \cdots + x_n y_n + (-x_{n+1})(y_{n+1})$$



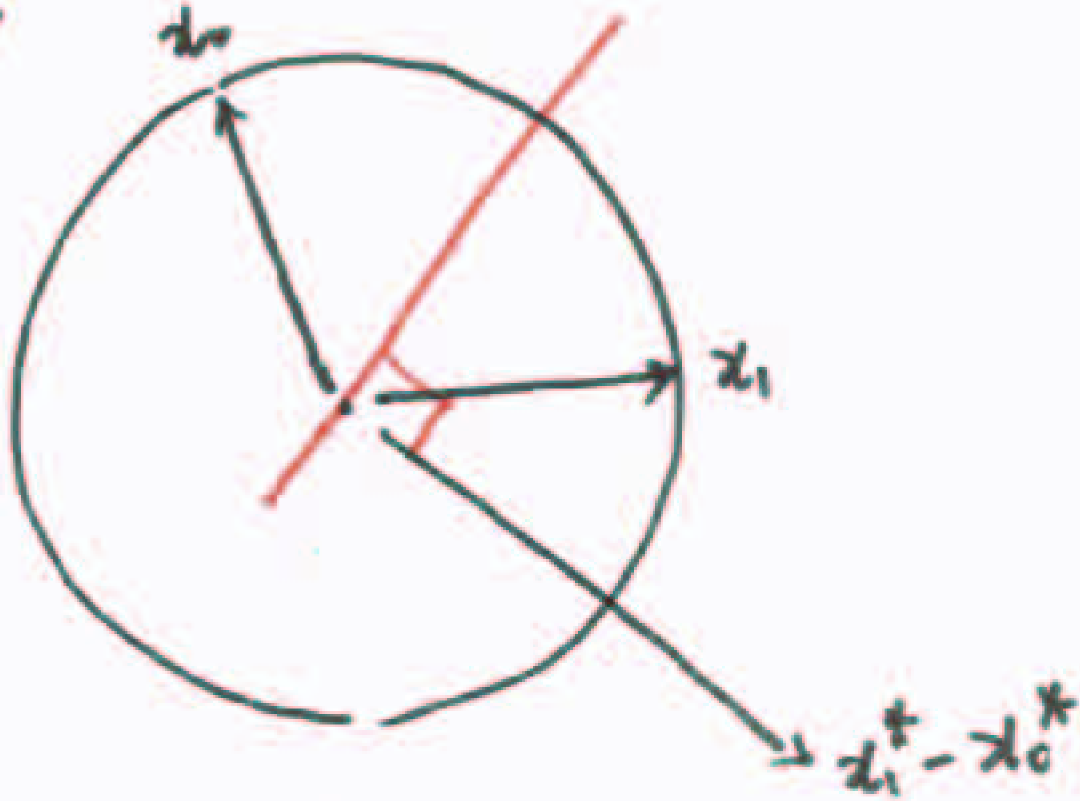
Thus,

$$
\begin{aligned}
&Bis(x_0, x_1) \\
&= \{ y \mid d_{\mathbb{H}^n}(x_0, y) = d_{\mathbb{H}^n}(x_1, y) \} \\
&= \{ y \mid h(x_0, y) = h(x_1, y) \} \\
&= \{ y \mid \langle x_0^*, y \rangle_E = \langle x_1^*, y \rangle_E \} \\
&= \{ y \mid \langle x_1^* - x_0^*, y \rangle_E = 0 \} \\
&= (x_1^* - x_0^*)^{\perp} \cap \mathbb{H}^n.
\end{aligned}
$$

$\therefore$ $Bis(x_0, x_1)$ is totally geodesic.



Similarly,



Question:

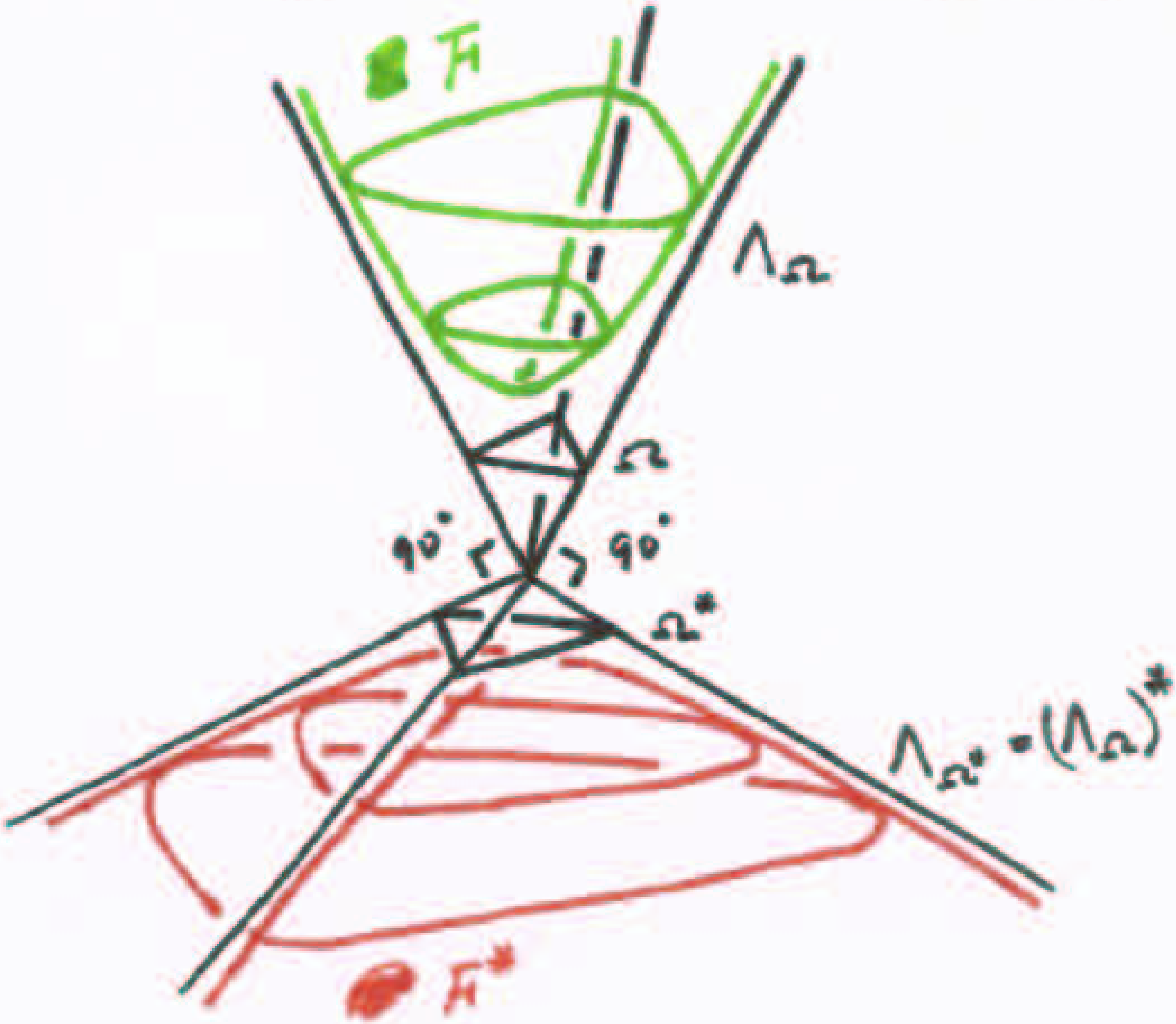
For any properly convex $\Omega \subset S^n$, do we have an analogue A of L so that

1. $\Lambda_\Omega = \{ x \mid A(x,x) < 0 \}$
2. $\{ x \mid A(x,x) = -1 \}$ resembles the hyperboloid
3. We can define $x \mapsto x^*$ s.t.

$$A(x,y) = \langle x^*, y \rangle_E \quad ?$$

Yes, we can construct such A using hyperbolic affine spheres.

... and they come as a dual pair.



- There is a natural correspondence

$$\Omega \longleftrightarrow \Omega^*$$
$$\Lambda_\Omega \longleftrightarrow \Lambda_{\Omega^*}$$
$$\mathcal{H} \longleftrightarrow \mathcal{H}^*$$
$$x \longleftrightarrow x^*$$

Conclusion.

Def For $x, y \in \mathcal{F} \subset \Lambda_\Omega$, define

$$A(x, y) = \langle x^*, y \rangle_E.$$

Thm (-) Let $M = \Omega/\Gamma$ be as before. The action $\Gamma \curvearrowright \Omega$ admits a convex, locally finite fundamental domain.

Pf) We shall show:

1. $A(gx, gy) = A(x, y)$. $\forall g \in \Gamma$. $\forall x, y \in \mathcal{F}$
2. $Bis(x_0, x_1) = (x_1^* - x_0^*)^\perp \cap \mathcal{F}$.
3. If $x_i \to \infty$, $\{Bis(x_0, x_i)\}$ is loc. finite
4. $\mathcal{F} = \{x \mid A(x, x) = -1\}$.

Then, the Dirichlet domain w.r.t. A is a convex, loc. finite fundamental domain

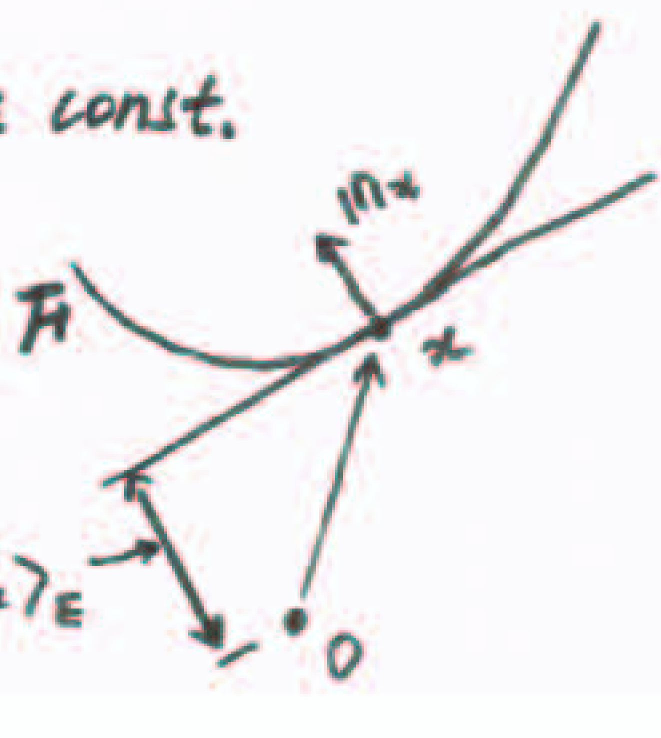
Proper Affine Spheres.

Assume :

- $\mathcal{H} \subset \mathbb{R}^{n+1} - \{0\}$. immersed hypersurface.
- $\mathcal{H}$. locally strongly convex.
- $\mathfrak{n}_x$, unit normal into the convex side
- K_x, Gaussian curvature.

Def $\mathcal{H}$ is a proper affine sphere of affine mean curvature λ with center 0, provided

$$-\frac{K_x^{\frac{1}{n+2}}}{\langle x, \mathfrak{n}_x \rangle_E} \equiv \lambda \equiv \text{const.}$$



Def. If $\lambda > 0$, $\mathcal{H}$ is elliptic
$\lambda < 0$ hyperbolic.

Thm (Tzitzeica, '07) 1907.

$$\left[-\frac{K_x^{\frac{1}{n+2}}}{\langle x, m_x \rangle_E} \equiv \lambda \equiv \text{const.} \right]$$

is an equi-affine property, i.e., invariant under $SL^{\pm}_{n+1}\mathbb{R}$.

Thm (Cheng–Yau, '86).
(Calabi, Nirenberg, Li, Gigena, Sasaki)

$$\left[\begin{array}{l}\text{Properly convex} \\ \text{cones in } \mathbb{R}^{n+1}\end{array}\right] \longleftrightarrow \left[\begin{array}{l}\text{Complete hyp.} \\ \text{affine spheres} \\ \text{of curvature} = \lambda \\ \text{with center } 0.\end{array}\right]$$

Rem. Calabi conjectured this in '72.

Def (Conormal map $x \mapsto x^*$)

For $\mathcal{H}$, hyp. affine sphere of affine mean curvature $\lambda = -1$ with center 0, define

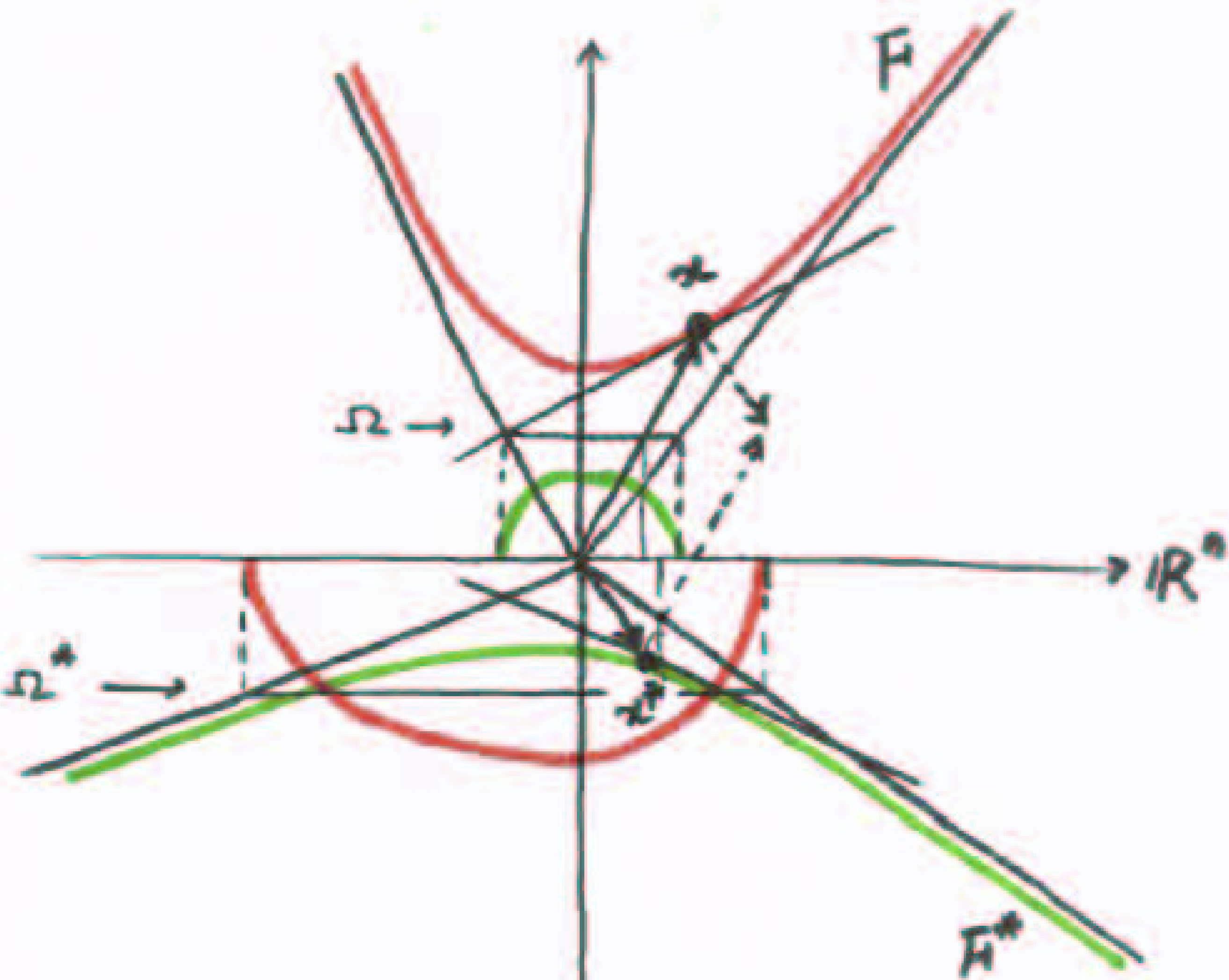
$$x^* = -K_x^{-\frac{1}{n+2}} \cdot \mathrm{n}_x ,$$

(so that we then have
$\langle x^*, T_x \mathcal{H} \rangle_E = 0, \quad \langle x^*, x \rangle_E = -1$)

4.

Thm (Calabi, '50s (?)).

$\mathcal{H}^* = \{ x^* \mid x \in \mathcal{H} \}$ is also a hyp. affine sphere of $\lambda = -1$, center $= 0$.



$F = \{ (a, f(a)) \mid a \in \mathbb{R}^n \}.$

$g(b) := \langle b, a \rangle_E - f(a),$ $\quad b = (\operatorname{grad} f)(a) \in \Omega^*$

Legendre transformation.

$$\begin{cases} \det(g_{ij}) = (\lambda g)^{-(n+2)} \\ g|_{\partial\Omega^*} \equiv 0 \end{cases}$$

(Cheng-Yau solved this boundary value prob.)

1. $A(gx, gy) = A(x, y)$

Let $g \in \Gamma < \mathrm{Aut}(\Omega) < SL^{\pm}_{n+1}\mathbb{R}$.

Γ acts on Λ_Ω by $g \cdot x$

Λ_{Ω^*} $\quad (g^*)^{-1} \cdot x^*$

By the uniqueness of Cheng-Yau,

Γ acts on $\mathcal{H}$ & $\mathcal{H}^*$ in the same way.

Now, for $x, y \in \mathcal{H}$,

$$\begin{aligned} A(gx, gy) &= \langle (gx)^*, gy \rangle_E \\ &= \langle (g^*)^{-1} x^*, gy \rangle_E \\ &= \langle x^*, g^{-1} g y \rangle_E \\ &= \langle x^*, y \rangle_E \\ &= A(x, y). \end{aligned}$$

3. For $x_i \to \infty$, $\{Bis(x_0, x_i)\}$ is locally finite.

If not, $Bis(x_0, x_i) \longrightarrow H$ and $H \cap \bar{F} \neq \phi$

Then, $x_i^* - x_0^* \longrightarrow H^\perp$

i.e. $x_i^* \longrightarrow x_0^* + H^\perp$.

$\nleftrightarrow$ to $x_i \longrightarrow \infty$.

End of proof $\blacksquare$

Further Questions.

1. $A(x, y)$ was defined on $\bar{H} \times \bar{H}$.

   By scaling, can be extended to $\Lambda_\Omega \times \mathbb{R}^{n+1}$

   Can it be further extended to $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$?

   i.e. what is x^* for $x \notin \Lambda_\Omega$?

2. Relation between $A(x, y) = \langle x^*, y \rangle_E$

   and $\bar{A}(x, y) = \langle x, y^* \rangle_E$?

   Can we somehow average the two

   so as to get a symmetric one ?

3. We have $A(x, y) \leq -1$, $\forall x, y$.

   What are $\log(-A)$ or $\cosh^{-1}(-A)$?

   Relation to the Hilbert metric d_Ω ?

- END -