

# A characterization of cones in the projective space

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# Affine domains and projective domains

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Affine domains can be considered as projective domains by the following equivariant embedding :

$$(i, \rho) : (\mathbb{E}^n, \text{Aff}(n)) \rightarrow (\mathbb{RP}^n, \text{PGL}(n+1, \mathbb{R}))$$

$$i(x_1, \dots, x_n) = [x_1, \dots, x_n, 1]$$

$$\rho(A, a) = \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$$

## Cones in the projective space

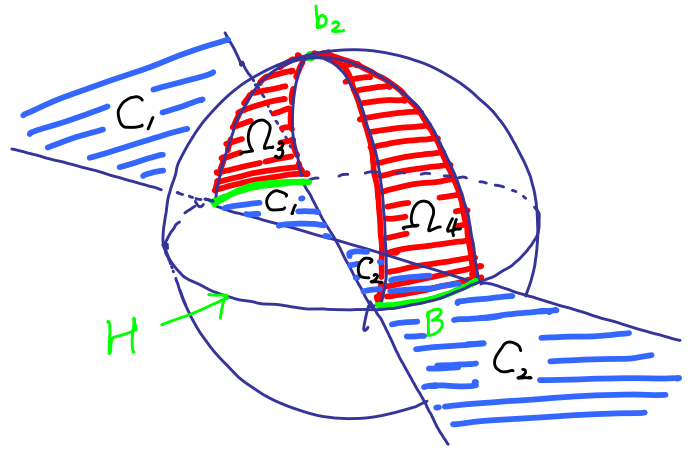
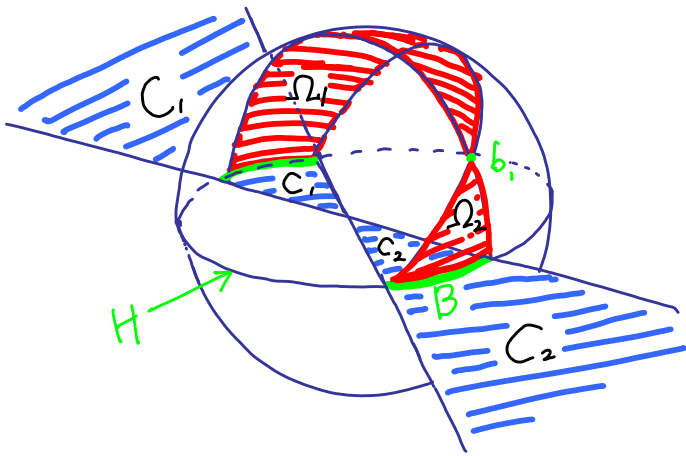
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**Definition 1.**  $\Omega$  : a domain in  $\mathbb{RP}^n$

$B$  : a domain of a hyperplane  $H$  of  $\mathbb{RP}^n$

- $C(B)$  : a *cone over  $B$*   
a cone with the infinite boundary  $\overline{B}$  in the affine space  $\mathbb{A}^n = \mathbb{RP}^n \setminus H$ ,
- $\{b\} \vee B$  : a *cone over  $B$*  with a cone point  $b$ .

- (i)  $C(B)$  is projectively equivalent to each component of  $\pi^{-1}(B)$ , where  $\pi : \mathbb{R}^n \rightarrow H \simeq \mathbb{RP}^{n-1}$ ,
- (ii) There are two cones over  $B$  with a cone point  $b$ ,
- (iii)  $C(B)$  is well-defined up to projective equivalence (not depending on the cone point)
- (iv)  $\{b\} \vee B = \{b\} \dot{+} B$ , if  $B$  is properly convex.



$\Omega_1, \Omega_2, \Omega_3, \Omega_4$  : Cones over  $B$   
 projectively equivalent to  $C_1$  and  $C_2$

$$C_1 \cup C_2 = \pi^{-1}(B), \quad \pi = \mathbb{R}^2 \rightarrow H \cong \mathbb{R}P^1$$

## convex sums

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- A properly convex domain  $\Omega$  is called a *convex sum* of its faces  $F_1$  and  $F_2$ , which will be denoted by

$$\Omega = F_1 \dot{+} F_2,$$

if it is the interior of the convex hull of  $\overline{F_1} \cup \overline{F_2}$  when we consider  $\Omega$  as a bounded set in an affine space  $\mathbb{A}^n$  in  $\mathbb{RP}^n$ , i.e., it is the union of all open line segments joining points in  $F_1$  to points in  $F_2$ .

- Note that if the dimensions of  $F_1$ ,  $F_2$  and  $\Omega$  are  $k_1$ ,  $k_2$  and  $n$  respectively, then  $n = k_1 + k_2 + 1$ .

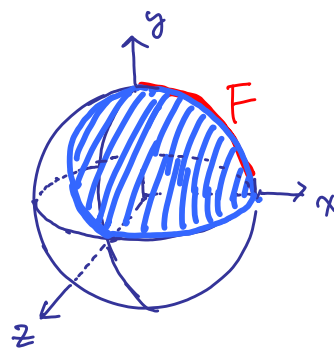
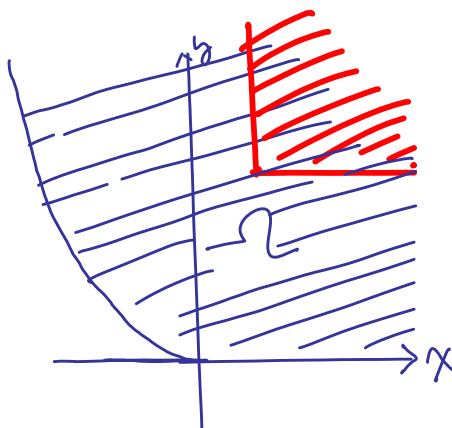
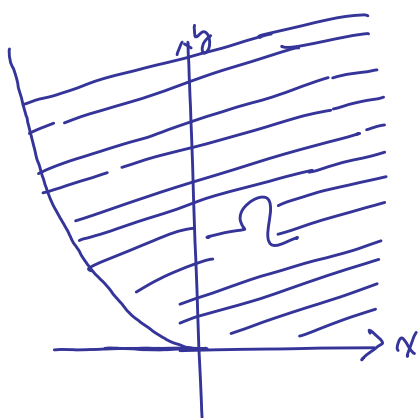
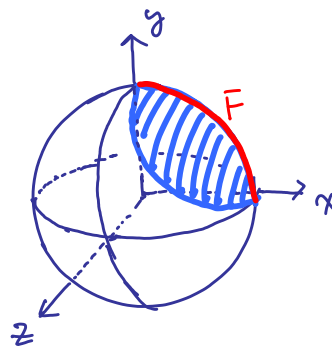
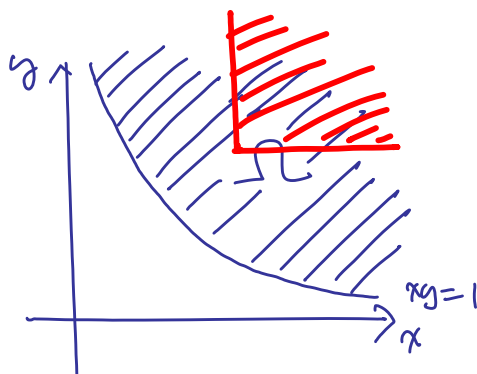
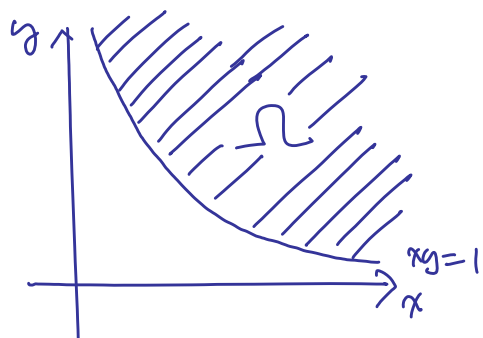


## Previous results

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- **Vinberg**(1963) classified all homogeneous convex cones algebraically,
- **Kuiper**(1953) classified 2-dimensional quasi-homogeneous convex domains while he was classifying convex compact projective surfaces,
- **Vey**(1970)  
Any quasi-homogeneous properly convex affine domain  $\Omega$  is a cone if it contains an open cone,
- **Benzécri**(1960)  
Any quasi-homogeneous properly convex projective domain with a face  $F$  of codimension 1 is the convex sum of  $F$  and a point in the boundary.
- **Benoist** (2000s) has been studying divisible convex domains and found many interesting examples.

**Key** : Any quasi-homogeneous properly convex affine domain is a cone if it contains an open cone.



$\Omega \subset \mathbb{R}^n$  has an open cone.

$\Downarrow$

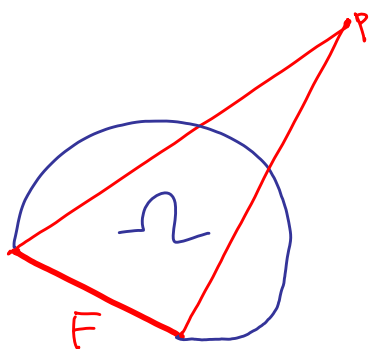
$\Omega \subset \mathbb{RP}^n$  has a face of codim 1.



Benzécri :  $\Omega$ : quasi-homogeneous properly convex domain

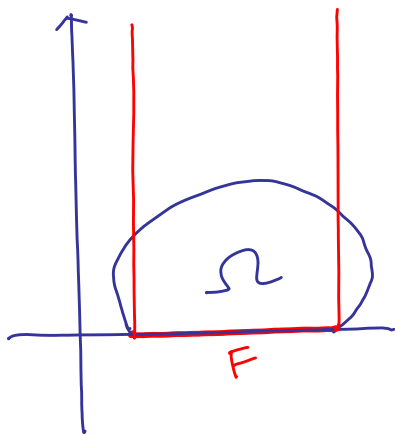
$F$ : face of  $\Omega$  with codim 1.

$\Rightarrow \Omega = F + \{b\}$  for some  $b \in \partial\Omega$



$\exists g \in \text{PGL}(3, \mathbb{R})$  s.t.

$g^n(\Omega) \rightarrow F + \{P\}$



$g = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$g^n(\Omega) \rightarrow F \times \mathbb{R}^+ \quad (\text{in } \mathbb{R}^2)$

$F + \{P\} \quad (\text{in } \mathbb{RP}^2)$

By stability of quasi-homogeneous properly convex domains,

$\Omega$  is projectively equivalent to  $F + \{P\}$ .

## Convex case I

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**Thm 1.**  $\Omega \subset \mathbb{RP}^n$ : properly convex domain

$F$  :  $(n - 1)$ -dimensional face of  $\Omega$

$\text{Aut}_{\text{proj}}(\Omega)x$  accumulates to a point in  $F, x \in \Omega$

Then

$$\Omega = \{\xi\} \dot{+} F, \quad \xi \in \partial\Omega.$$

*Proof.*

$\exists x \in \Omega, \{g_i\} \subset \text{Aut}_{\text{proj}}(\Omega)$  s.t.  $\lim_{i \rightarrow \infty} g_i(x) = p \in F$ .

$\Rightarrow g = \lim_{i \rightarrow \infty} g_i, \text{Ran}(g) = \langle F \rangle, g(\Omega) = F$   
and  $\text{Ker}(g) = \{z\}$  is an extreme point.

**Case 1:**  $z \notin \langle F \rangle$

**Case 2 :**  $z \in \langle F \rangle$

**Case 1:**  $z \notin \langle F \rangle$

$$\lim_{i \rightarrow \infty} g_i(\overline{F}) = g(\overline{F}) = \overline{F}$$

$$\Rightarrow g(\{z\} \dot{+} F) = F, g(\{z\} \dot{+} (\langle F \rangle \setminus \overline{F})) = \langle F \rangle \setminus \overline{F}$$

$\Rightarrow \Omega = \{z\} \dot{+} F$ , one of two convex sum by connectedness of  $\Omega$ .

**Case 2 :**  $z \in \langle F \rangle$

$$E = \{b \in \partial\Omega \mid \overline{bz} \cap \Omega \neq \emptyset\}$$

$\Rightarrow E$  is an  $(n - 1)$ -dimensional face of  $\Omega$ .

$$\Rightarrow \{z\} \dot{+} E = \Omega$$

$$\Rightarrow \Omega = g_k(\Omega) = \{g_k(z)\} \dot{+} g_k(E) = \{g_k(z)\} \dot{+} F$$

for some  $k$ , since  $g_i(E)$  uniformly converges to  $g(E) \subset F$ . □

## Convex case II

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**Coro 1.** *Let  $\Omega$  be a convex domain in  $\mathbb{R}^n$  and  $F$  an  $(n - 1)$ -dimensional face of  $\Omega$ . Suppose that there is a sequence  $\{g_i\}$  of affine transformations which preserve  $\Omega$  and a point  $x$  in the interior of  $\Omega$  such that  $\{g_i(x)\}$  accumulates to an interior point of  $F$ . Then*

$$\Omega = \mathbb{R}^+ \times F.$$

## locally flat boundary point

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We say  $\partial\Omega$  is *locally flat* at  $p$  if there is a hyperplane  $H$  and an open ball  $B_p$  centered at  $p$  such that  $\Omega \cap B_p$  is an open half ball with  $H \cap B_p \subset \partial\Omega$ .

**Definition 3.** Let  $\Omega$  be a domain in  $\mathbb{R}\mathbb{P}^n$ .

- (i) we say that  $\Omega$  has a *flat boundary piece*  $P$  if  $P^0 \neq \emptyset$  in  $H$  and  $\partial\Omega$  is locally flat at each  $p \in P^0$ ,
- (ii) we say that  $\Omega$  has a *strongly flat boundary piece*  $P$  if  $P$  is a flat boundary piece of  $\Omega$  with  $\langle P \rangle = H$  and there is an open neighborhood  $U$  of  $P$  such that  $\Omega \cap U$  is contained in an open half space  $H^+$  with boundary  $H$ .

## counter examples

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The existence of an accumulation point in the flat boundary piece is not a sufficient condition for being a cone.

$$(i) \Omega = \Omega_1 \cup \Omega_2$$

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 \mid x > 1\}$$

$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1, y > 0\},$$

$$g_n = \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}, P = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, y = 0\}.$$

- $P$  is a flat boundary piece of  $\Omega$ ,
- $\lim_{n \rightarrow \infty} g_n(1/2, 1) = (1/2, 0) \in P^0$ ,
- $\Omega$  is not a cone.

$$(ii) \Omega = \Omega_1 \cup \Omega_2$$

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 \mid x \leq 0, y > 0\}$$

$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 \mid x > 0, 0 < y < 1/x\},$$

$$g_n = \begin{pmatrix} 2^n & 0 \\ 0 & 1/2^n \end{pmatrix}, P = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}.$$

- $P$  is a flat boundary piece of  $\Omega$ ,
- $\lim_{n \rightarrow \infty} g_n(0, y) = (0, 0) \in P^0$ ,
- $\Omega$  is not a cone.

## Domains with a flat boundary piece

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**Thm 2.** *Let  $\Omega$  be a domain with a flat boundary piece  $P$  satisfying*

- (i)  $P$  is a component of  $\langle P \rangle \cap \overline{D}$ ,*
- (ii)  $P$  has no complete line.*

*Then  $\Omega = C(B)$  iff there is an accumulation point  $p \in P^0$  under the action of  $\text{Aut}(\Omega)$ .*

## Quasi-homogeneous domains

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**Thm 3.** *Let  $\Omega$  be a quasi-homogeneous affine domain with a flat boundary piece  $P$  satisfying*

- (i)  $P$  is a component of  $\langle P \rangle \cap \overline{D}$ ,*
- (ii)  $P$  has no complete line.*

*Then  $\Omega = \mathbb{R}^+ \times P^0$ , which is projectively equivalent to  $C(P^0)$ .*

- In convex case,  
Theorem 1  $\implies$  Vey(1970), Benzécri(1960),  
( $\because$  every point in any face of properly convex domain is an accumulation point.)
- A quasi-homogeneous domain is stable. ( $\times$ )
- $\exists g_n$  such that  $\lim_{n \rightarrow \infty} g_n(\Omega) = C(P^0)$ . ( $\times$ )

So we cannot apply Benzécri's idea to non-convex case.

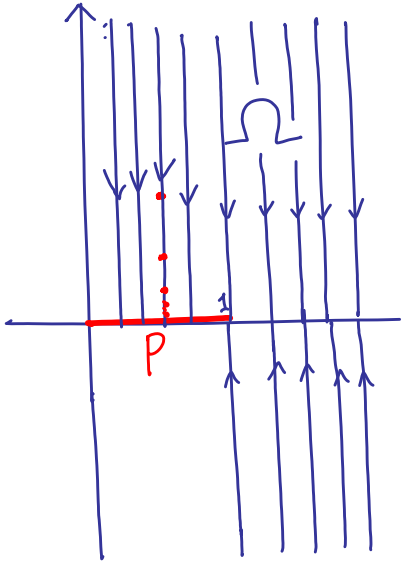
*Proof.* (i) Show that  $\exists$  a sequence  $g_n$  and  $x \in \Omega$  such that  $g_n(x)$  converges to a point in  $P^0$ .

(ii) Apply Theorem 2.

□



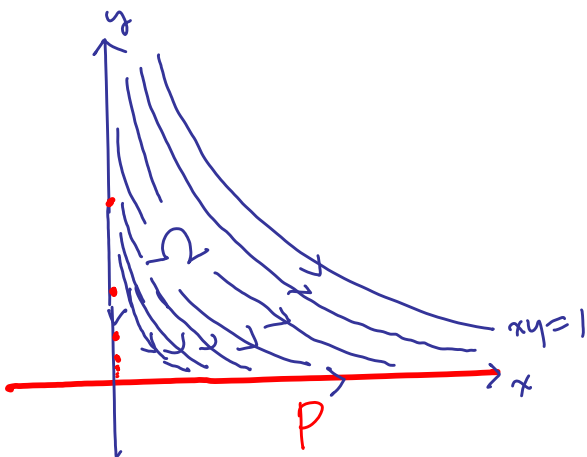
Counter example (i)



$$g_n = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$$

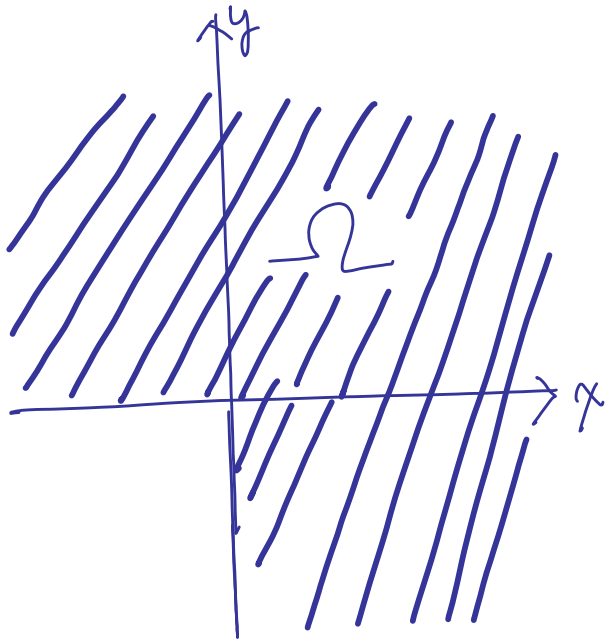
$P$  is not a component of  $\langle P \rangle \cap \bar{\Omega}$ .

Counter example (ii)



$$g_n = \begin{pmatrix} 2^n & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix}$$

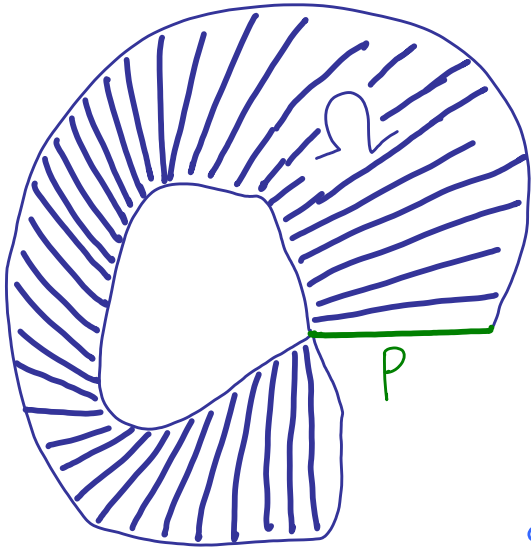
$P$  has a complete line.



$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$$

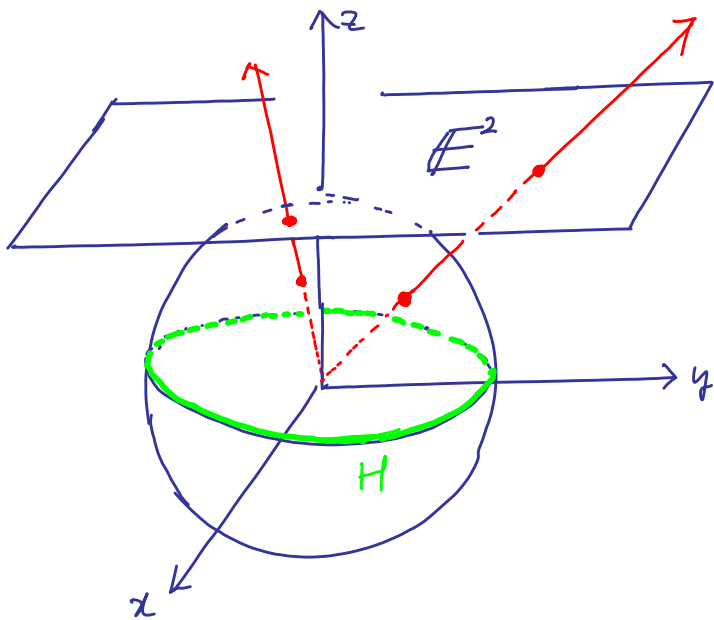
$\Omega$  is not a cone in our sense.

But  $\Omega$  is a cone in a vector space  $\mathbb{R}^2$ .

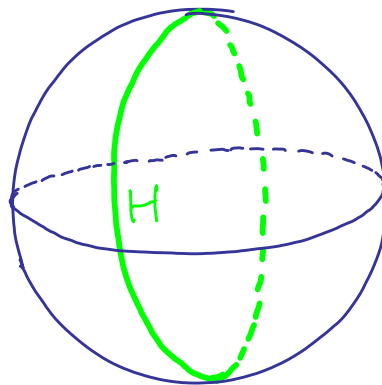
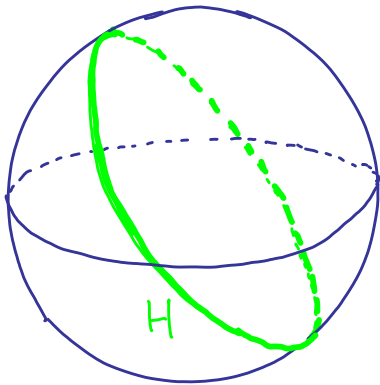


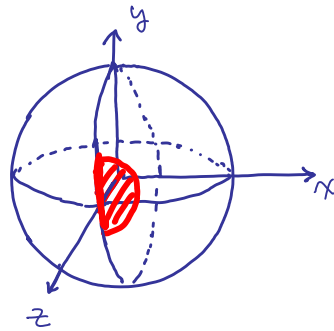
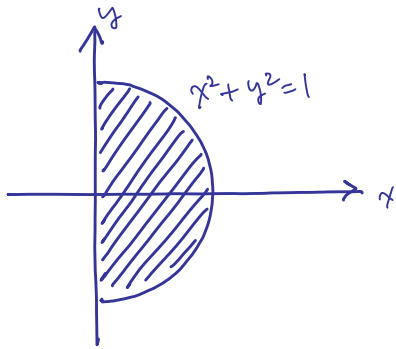
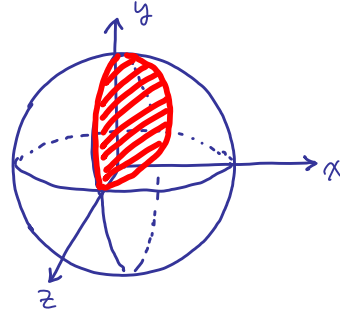
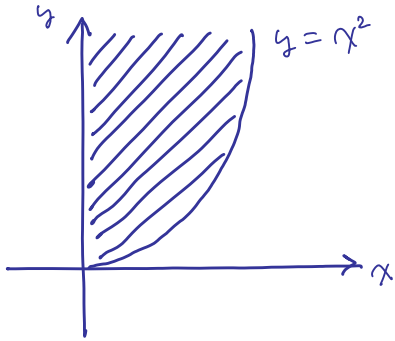
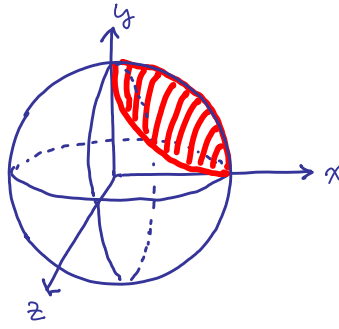
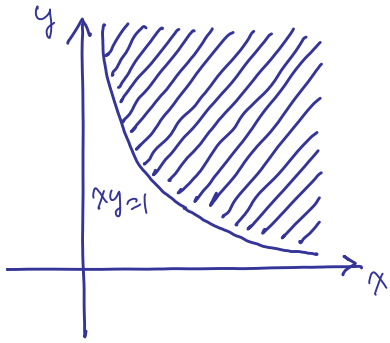
$P$  is a flat boundary piece of  $\Omega$ ,  
but not strongly flat.

$P$  is a component of  $\langle P \rangle \cap \bar{\Omega}$   
and  $P$  has no complete line.

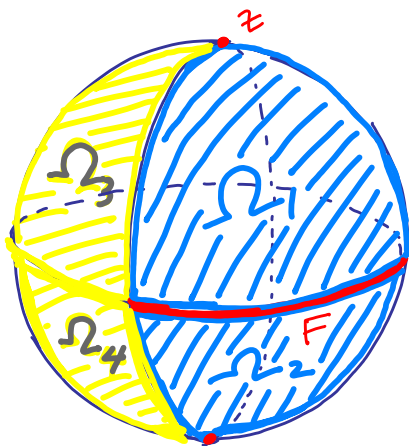


$$\mathbb{R}P^3 - H \cong \mathbb{E}^2.$$





Case 1.



$\Omega_1, \Omega_2 = \text{convex sums of } F \text{ and } \{z\}.$

$F + \{z\}$

$C(F), F \cup \{z\}$

$\Omega_3, \Omega_4 = (\langle F \rangle \setminus \bar{F}) \cup \{z\}.$

$$g(\Omega_1 \cup \Omega_2) = F.$$

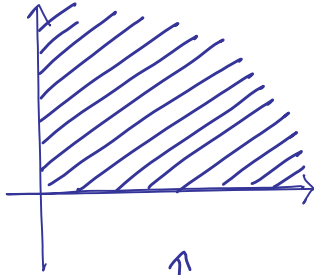
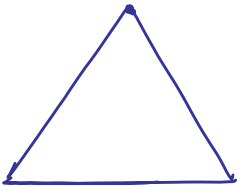
$$g(\Omega_3 \cup \Omega_4) = \langle F \rangle \setminus \bar{F}$$

$$g(\Omega) = F$$

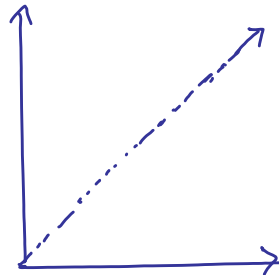
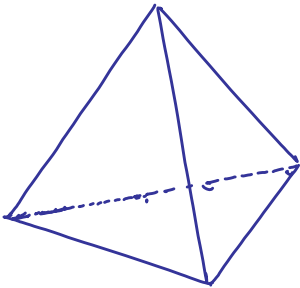
$\Rightarrow \Omega$  is either  $\Omega_1$  or  $\Omega_2$ .



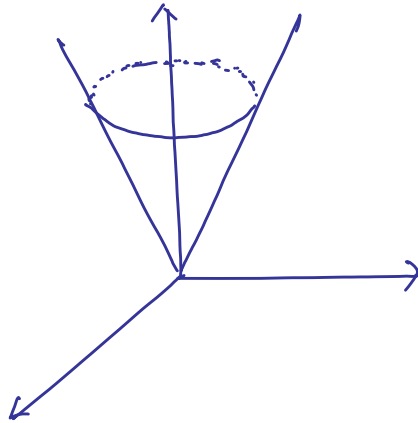
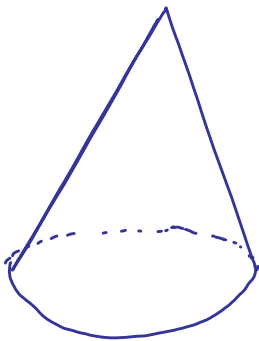
Cone over a point



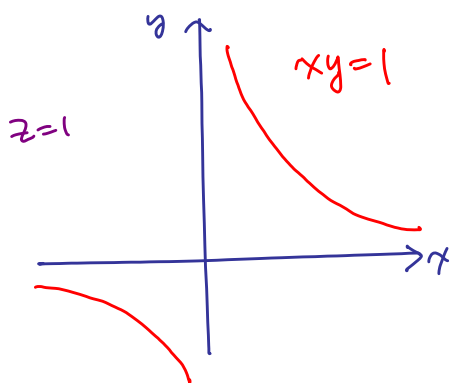
Cone over an interval



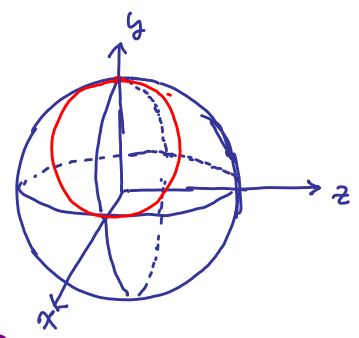
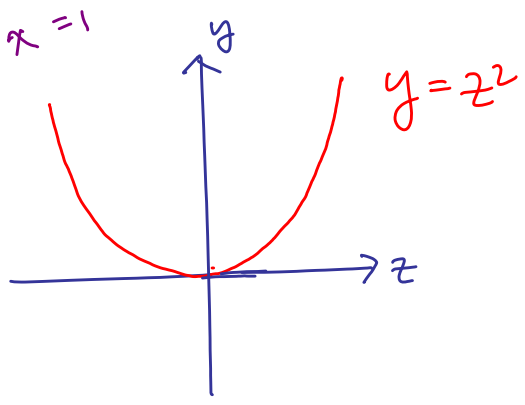
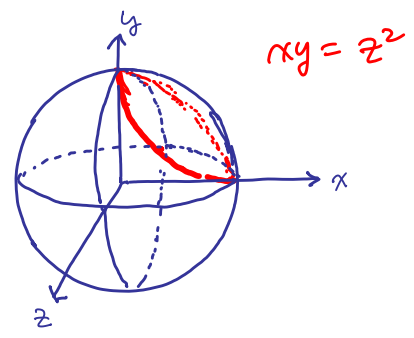
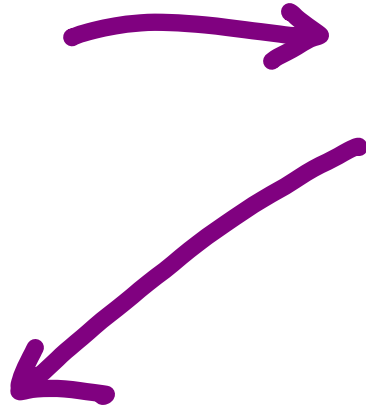
Cone over a triangle



Cone over a ball  
(Cone over a strictly convex domain)

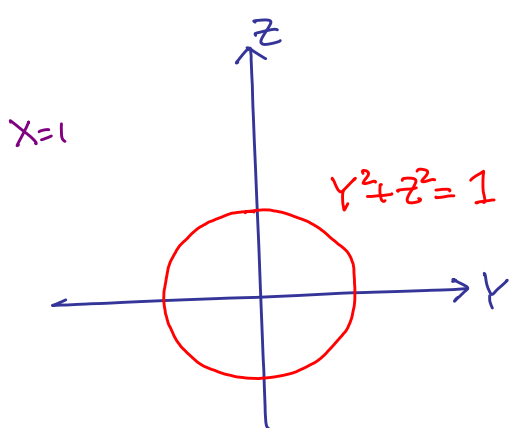
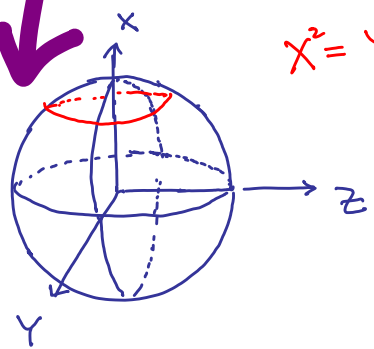
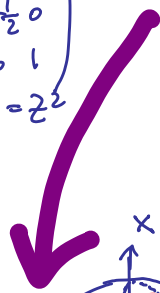


$$(a, \frac{1}{a}, 1) \longleftrightarrow [at, \frac{t}{a}, t]$$

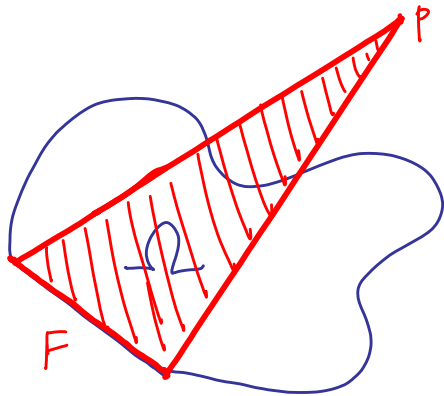


$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

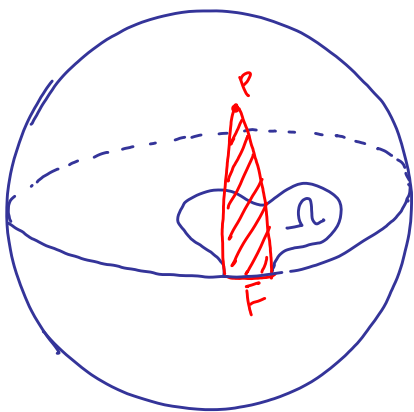
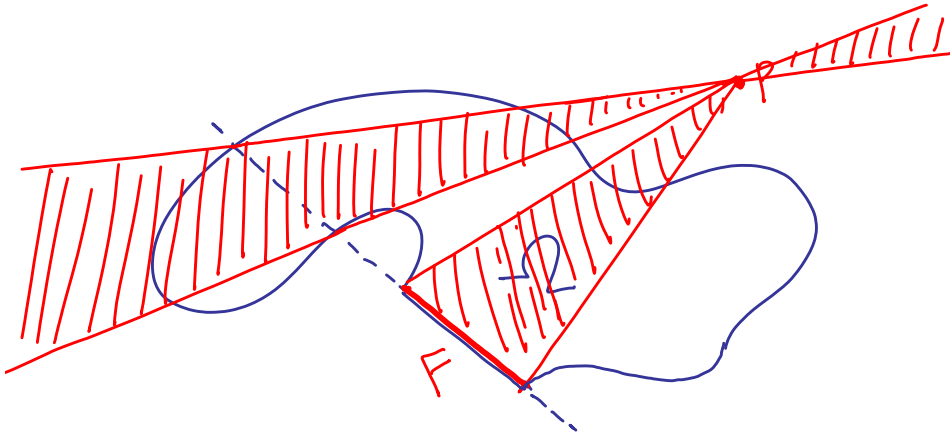
$$(X+Y)(X-Y) = Z^2$$



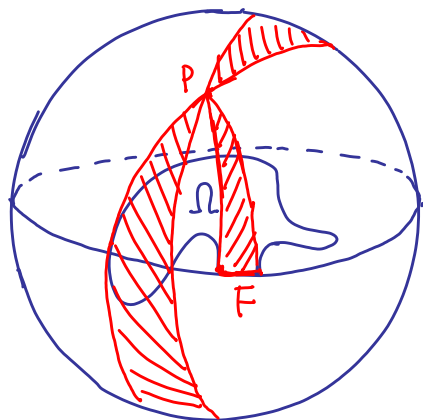




$\exists g_n \in PGL(3, \mathbb{R})$  s.t.  
 $g_n(\Omega) \rightarrow C(F)$



$g_n(\Omega) \rightarrow C(F)$



$g_n(\Omega) \not\rightarrow C(F)$