

A characterization of cones in the projective space

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Affine domains and projective domains

Affine domains can be considered as projective domains by the following equivariant embedding :

$$(i, \rho) : (\mathbb{E}^n, \text{Aff}(n)) \rightarrow (\mathbb{RP}^n, \text{PGL}(n+1, \mathbb{R}))$$

$$i(x_1, \dots, x_n) = [x_1, \dots, x_n, 1]$$

$$\rho(A, a) = \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}$$

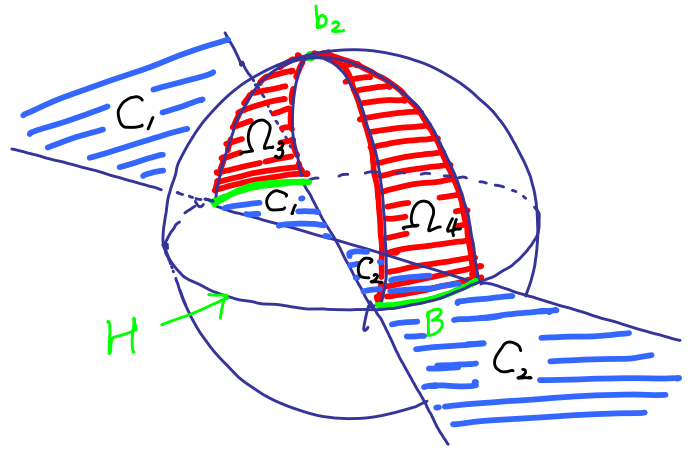
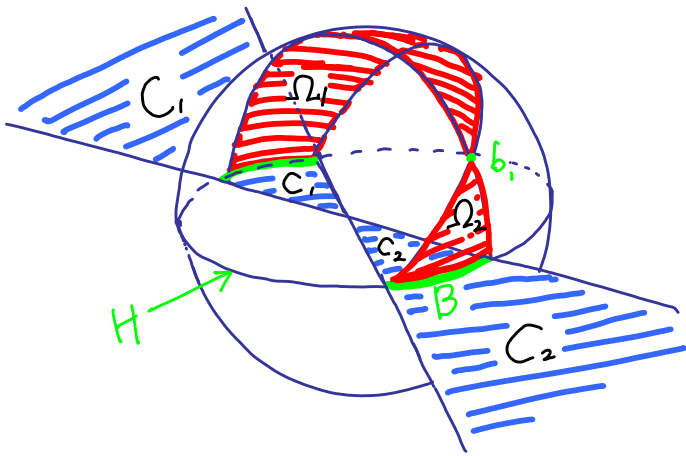
Cones in the projective space

Definition 1. Ω : a domain in \mathbb{RP}^n

B : a domain of a hyperplane H of \mathbb{RP}^n

- $C(B)$: a *cone over B*
a cone with the infinite boundary \overline{B} in the affine space $\mathbb{A}^n = \mathbb{RP}^n \setminus H$,
- $\{b\} \vee B$: a *cone over B* with a cone point b .

- (i) $C(B)$ is projectively equivalent to each component of $\pi^{-1}(B)$, where $\pi : \mathbb{R}^n \rightarrow H \simeq \mathbb{RP}^{n-1}$,
- (ii) There are two cones over B with a cone point b ,
- (iii) $C(B)$ is well-defined up to projective equivalence (not depending on the cone point)
- (iv) $\{b\} \vee B = \{b\} \dot{+} B$, if B is properly convex.



$\Omega_1, \Omega_2, \Omega_3, \Omega_4$: Cones over B
 projectively equivalent to C_1 and C_2

$$C_1 \cup C_2 = \pi^{-1}(B), \quad \pi = \mathbb{R}^2 \rightarrow H \cong \mathbb{R}P^1$$

convex sums

- A properly convex domain Ω is called a *convex sum* of its faces F_1 and F_2 , which will be denoted by

$$\Omega = F_1 \dot{+} F_2,$$

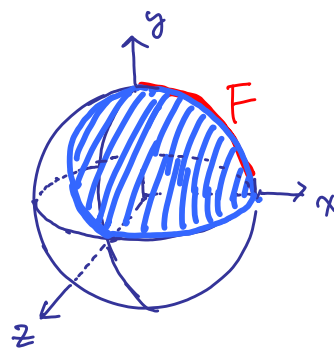
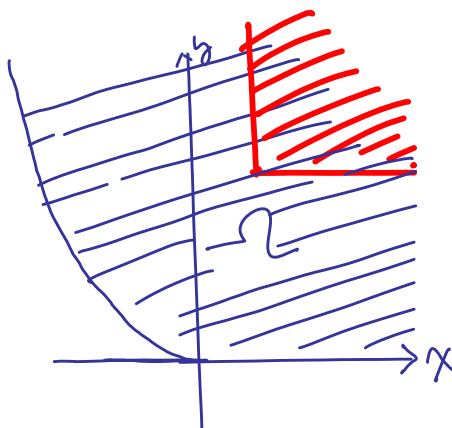
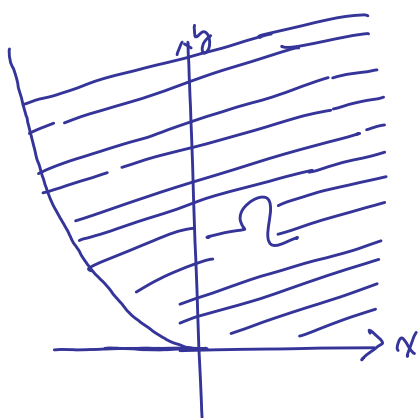
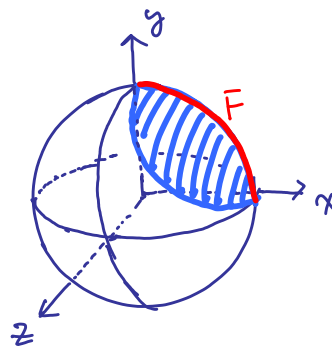
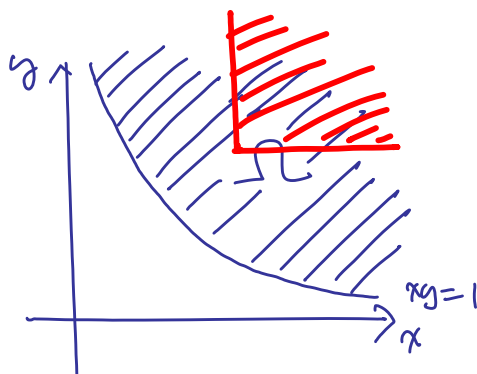
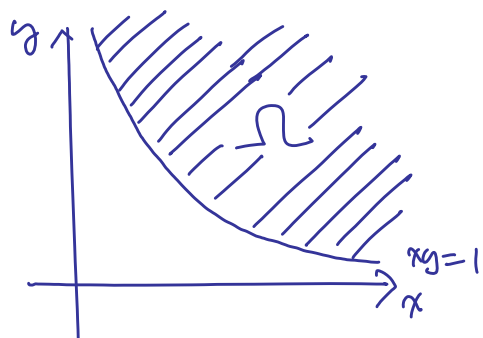
if it is the interior of the convex hull of $\overline{F_1} \cup \overline{F_2}$ when we consider Ω as a bounded set in an affine space \mathbb{A}^n in \mathbb{RP}^n , i.e., it is the union of all open line segments joining points in F_1 to points in F_2 .

- Note that if the dimensions of F_1 , F_2 and Ω are k_1 , k_2 and n respectively, then $n = k_1 + k_2 + 1$.

Previous results

- **Vinberg**(1963) classified all homogeneous convex cones algebraically,
- **Kuiper**(1953) classified 2-dimensional quasi-homogeneous convex domains while he was classifying convex compact projective surfaces,
- **Vey**(1970)
Any quasi-homogeneous properly convex affine domain Ω is a cone if it contains an open cone,
- **Benzécri**(1960)
Any quasi-homogeneous properly convex projective domain with a face F of codimension 1 is the convex sum of F and a point in the boundary.
- **Benoist** (2000s) has been studying divisible convex domains and found many interesting examples.

Key : Any quasi-homogeneous properly convex affine domain is a cone if it contains an open cone.



$\Omega \subset \mathbb{R}^n$ has an open cone.

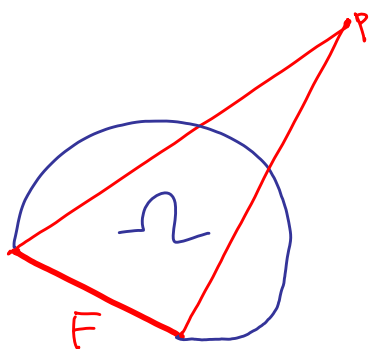
\Downarrow

$\Omega \subset \mathbb{RP}^n$ has a face of codim 1.

Benzécri : Ω : quasi-homogeneous properly convex domain

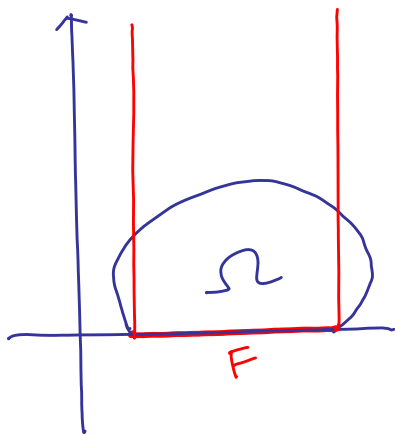
F : face of Ω with codim 1.

$\Rightarrow \Omega = F + \{b\}$ for some $b \in \partial\Omega$



$\exists g \in \text{PGL}(3, \mathbb{R})$ s.t.

$g^n(\Omega) \rightarrow F + \{P\}$



$g = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

$g^n(\Omega) \rightarrow F \times \mathbb{R}^+ \quad (\text{in } \mathbb{R}^2)$

$F + \{P\} \quad (\text{in } \mathbb{RP}^2)$

By stability of quasi-homogeneous properly convex domains,

Ω is projectively equivalent to $F + \{P\}$.

Convex case I

Thm 1. $\Omega \subset \mathbb{RP}^n$: properly convex domain

F : $(n - 1)$ -dimensional face of Ω

$\text{Aut}_{\text{proj}}(\Omega)x$ accumulates to a point in $F, x \in \Omega$

Then

$$\Omega = \{\xi\} \dot{+} F, \quad \xi \in \partial\Omega.$$

Proof.

$\exists x \in \Omega, \{g_i\} \subset \text{Aut}_{\text{proj}}(\Omega)$ s.t. $\lim_{i \rightarrow \infty} g_i(x) = p \in F$.

$\Rightarrow g = \lim_{i \rightarrow \infty} g_i, \text{Ran}(g) = \langle F \rangle, g(\Omega) = F$
and $\text{Ker}(g) = \{z\}$ is an extreme point.

Case 1: $z \notin \langle F \rangle$

Case 2 : $z \in \langle F \rangle$

Case 1: $z \notin \langle F \rangle$

$$\lim_{i \rightarrow \infty} g_i(\overline{F}) = g(\overline{F}) = \overline{F}$$

$$\Rightarrow g(\{z\} \dot{+} F) = F, g(\{z\} \dot{+} (\langle F \rangle \setminus \overline{F})) = \langle F \rangle \setminus \overline{F}$$

$\Rightarrow \Omega = \{z\} \dot{+} F$, one of two convex sum by connectedness of Ω .

Case 2 : $z \in \langle F \rangle$

$$E = \{b \in \partial\Omega \mid \overline{bz} \cap \Omega \neq \emptyset\}$$

$\Rightarrow E$ is an $(n - 1)$ -dimensional face of Ω .

$$\Rightarrow \{z\} \dot{+} E = \Omega$$

$$\Rightarrow \Omega = g_k(\Omega) = \{g_k(z)\} \dot{+} g_k(E) = \{g_k(z)\} \dot{+} F$$

for some k , since $g_i(E)$ uniformly converges to $g(E) \subset F$. □

Convex case II

Coro 1. *Let Ω be a convex domain in \mathbb{R}^n and F an $(n - 1)$ -dimensional face of Ω . Suppose that there is a sequence $\{g_i\}$ of affine transformations which preserve Ω and a point x in the interior of Ω such that $\{g_i(x)\}$ accumulates to an interior point of F . Then*

$$\Omega = \mathbb{R}^+ \times F.$$

locally flat boundary point

We say $\partial\Omega$ is *locally flat* at p if there is a hyperplane H and an open ball B_p centered at p such that $\Omega \cap B_p$ is an open half ball with $H \cap B_p \subset \partial\Omega$.

Definition 3. Let Ω be a domain in $\mathbb{R}\mathbb{P}^n$.

- (i) we say that Ω has a *flat boundary piece* P if $P^0 \neq \emptyset$ in H and $\partial\Omega$ is locally flat at each $p \in P^0$,
- (ii) we say that Ω has a *strongly flat boundary piece* P if P is a flat boundary piece of Ω with $\langle P \rangle = H$ and there is an open neighborhood U of P such that $\Omega \cap U$ is contained in an open half space H^+ with boundary H .

counter examples

The existence of an accumulation point in the flat boundary piece is not a sufficient condition for being a cone.

(i) $\Omega = \Omega_1 \cup \Omega_2$

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 \mid x > 1\}$$

$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1, y > 0\},$$

$$g_n = \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}, P = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, y = 0\}.$$

- P is a flat boundary piece of Ω ,
- $\lim_{n \rightarrow \infty} g_n(1/2, 1) = (1/2, 0) \in P^0$,
- Ω is not a cone.

(ii) $\Omega = \Omega_1 \cup \Omega_2$

$$\Omega_1 = \{(x, y) \in \mathbb{R}^2 \mid x \leq 0, y > 0\}$$

$$\Omega_2 = \{(x, y) \in \mathbb{R}^2 \mid x > 0, 0 < y < 1/x\},$$

$$g_n = \begin{pmatrix} 2^n & 0 \\ 0 & 1/2^n \end{pmatrix}, P = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}.$$

- P is a flat boundary piece of Ω ,
- $\lim_{n \rightarrow \infty} g_n(0, y) = (0, 0) \in P^0$,
- Ω is not a cone.

Domains with a flat boundary piece

Thm 2. *Let Ω be a domain with a flat boundary piece P satisfying*

- (i) P is a component of $\langle P \rangle \cap \overline{D}$,*
- (ii) P has no complete line.*

Then $\Omega = C(B)$ iff there is an accumulation point $p \in P^0$ under the action of $\text{Aut}(\Omega)$.

Quasi-homogeneous domains

Thm 3. *Let Ω be a quasi-homogeneous affine domain with a flat boundary piece P satisfying*

- (i) P is a component of $\langle P \rangle \cap \overline{D}$,*
- (ii) P has no complete line.*

Then $\Omega = \mathbb{R}^+ \times P^0$, which is projectively equivalent to $C(P^0)$.

- In convex case,
Theorem 1 \implies Vey(1970), Benzécri(1960),
(\because every point in any face of properly convex domain is an accumulation point.)
- A quasi-homogeneous domain is stable. (\times)
- $\exists g_n$ such that $\lim_{n \rightarrow \infty} g_n(\Omega) = C(P^0)$. (\times)

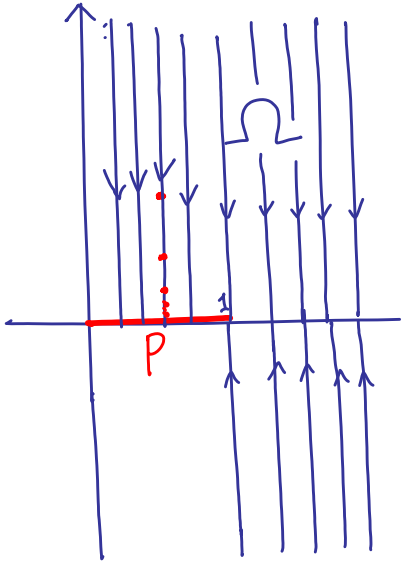
So we cannot apply Benzécri's idea to non-convex case.

Proof. (i) Show that \exists a sequence g_n and $x \in \Omega$ such that $g_n(x)$ converges to a point in P^0 .

(ii) Apply Theorem 2.

□

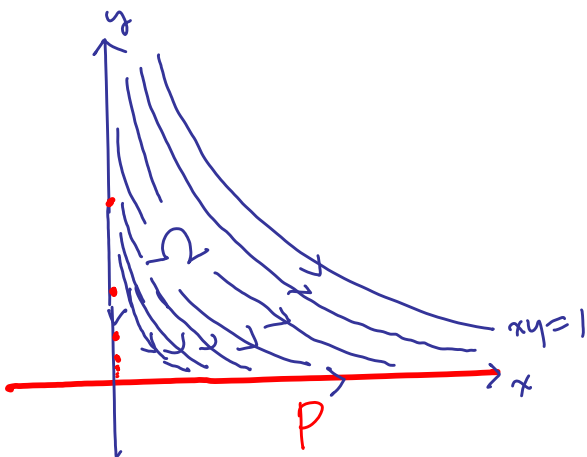
Counter example (i)



$$g_n = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$$

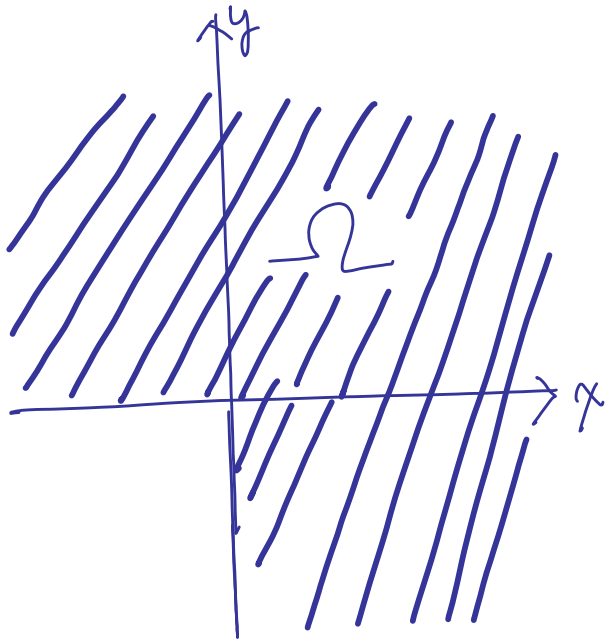
P is not a component of $\langle P \rangle \cap \bar{\Omega}$.

Counter example (ii)



$$g_n = \begin{pmatrix} 2^n & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix}$$

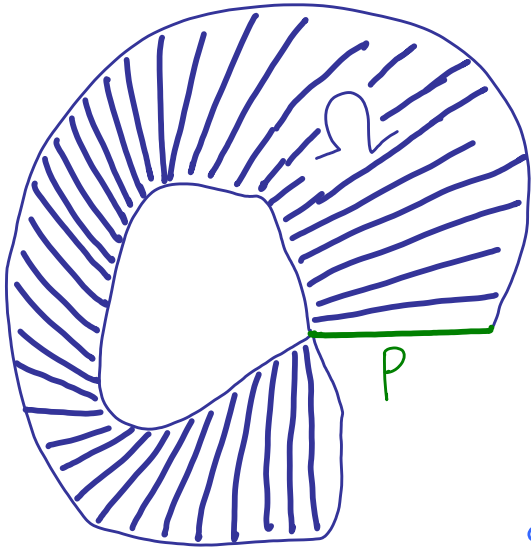
P has a complete line.



$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$$

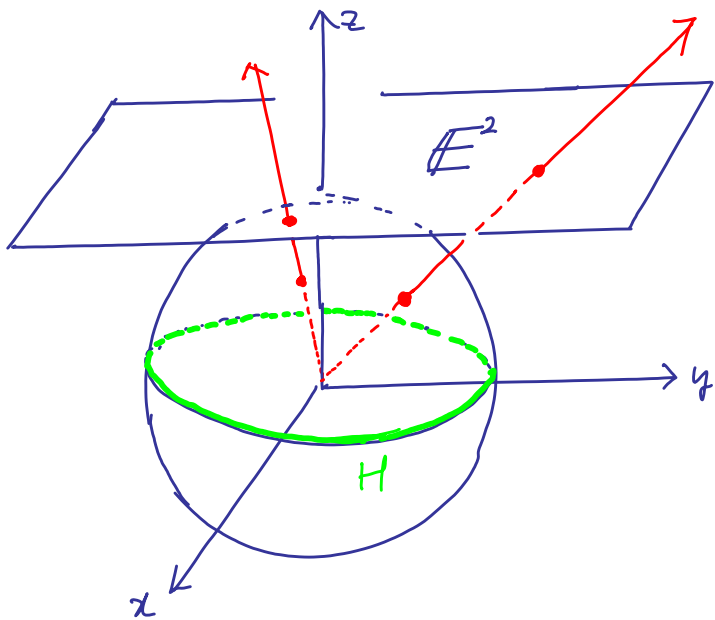
Ω is not a cone in our sense.

But Ω is a cone in a vector space \mathbb{R}^2 .

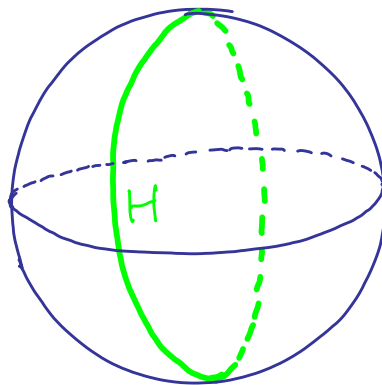
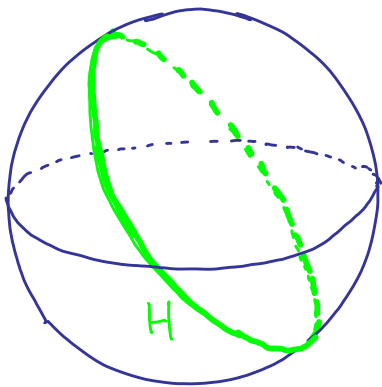


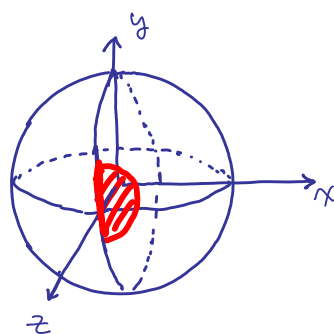
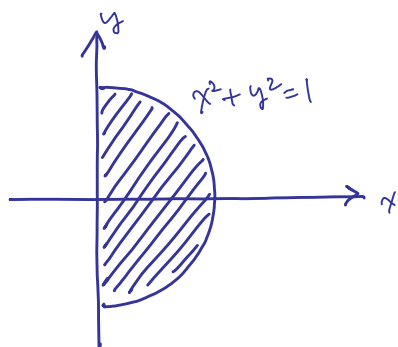
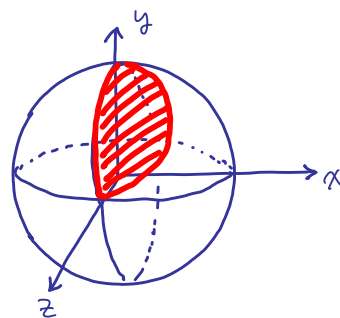
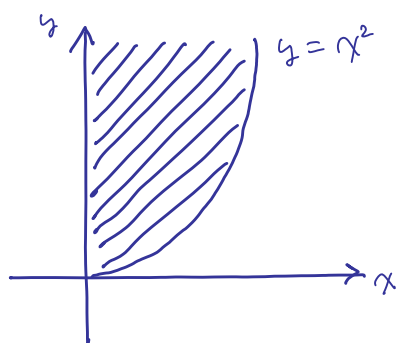
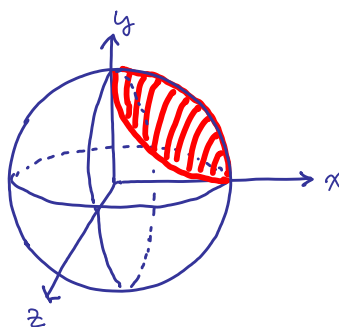
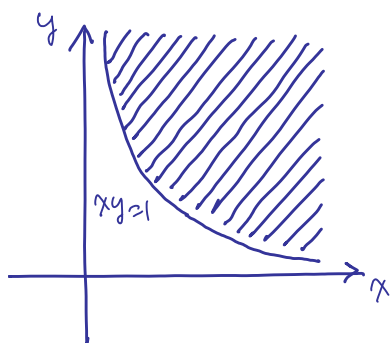
P is a flat boundary piece of Ω ,
but not strongly flat.

P is a component of $\langle P \rangle \cap \bar{\Omega}$
and P has no complete line.

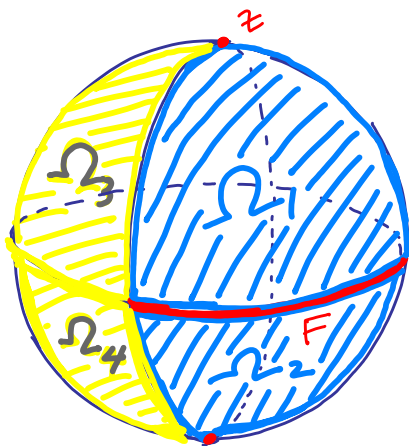


$$\mathbb{R}P^3 - H \cong \mathbb{E}^2.$$





Case 1.



$\Omega_1, \Omega_2 = \text{convex sums of } F \text{ and } \{z\}.$

$F + \{z\}$

$C(F), F \cup \{z\}$

$\Omega_3, \Omega_4 = (\langle F \rangle \setminus \bar{F}) \cup \{z\}.$

$$g(\Omega_1 \cup \Omega_2) = F.$$

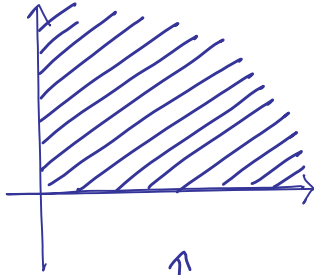
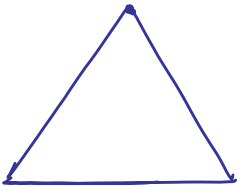
$$g(\Omega_3 \cup \Omega_4) = \langle F \rangle \setminus \bar{F}$$

$$g(\Omega) = F$$

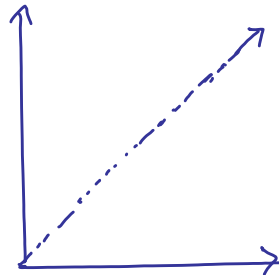
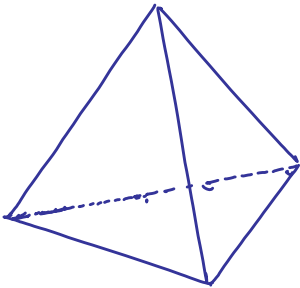
$\Rightarrow \Omega$ is either Ω_1 or Ω_2 .



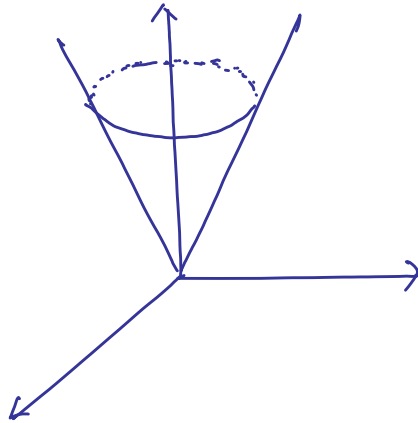
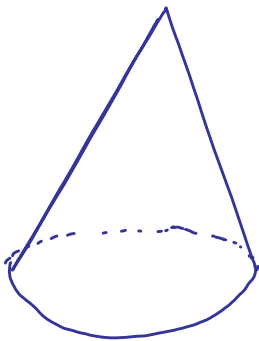
Cone over a point



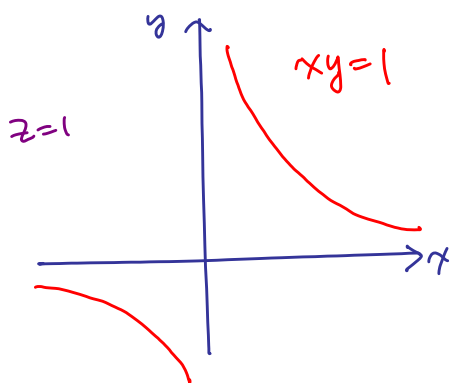
Cone over an interval



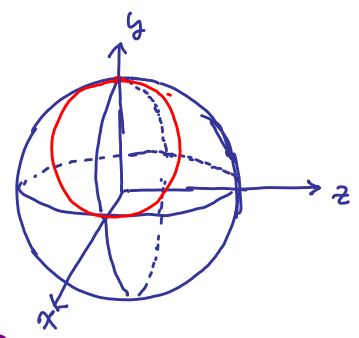
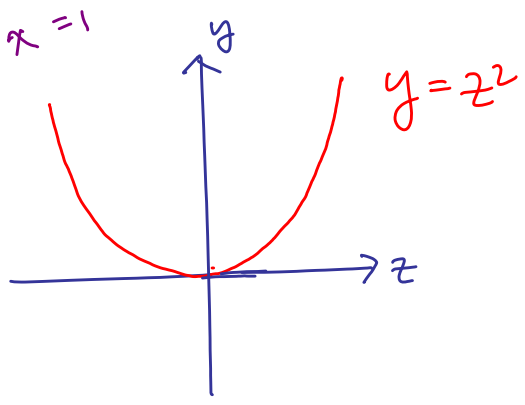
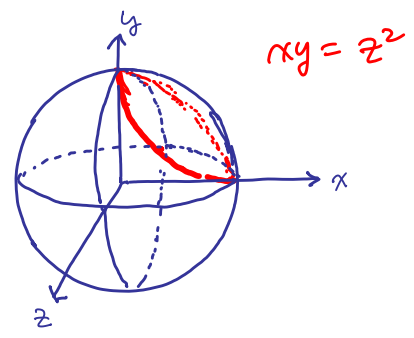
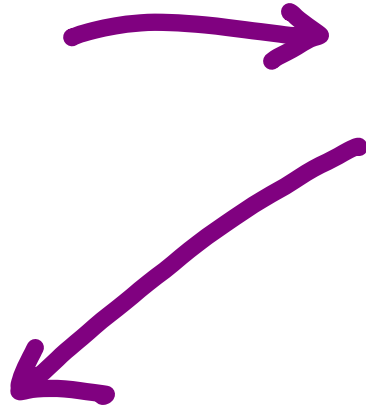
Cone over a triangle



Cone over a ball
(Cone over a strictly convex domain)

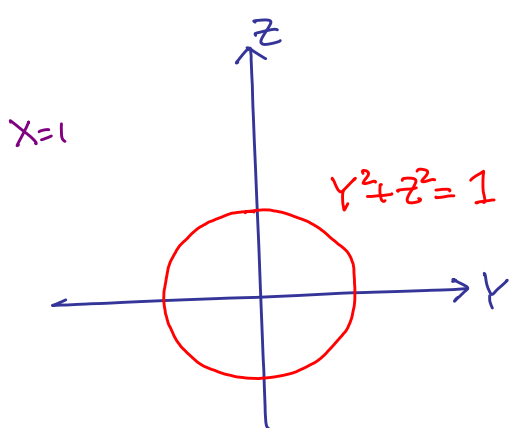
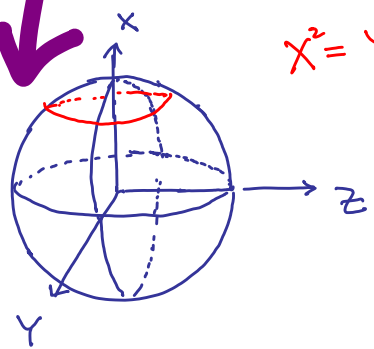
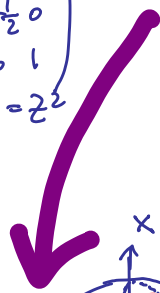


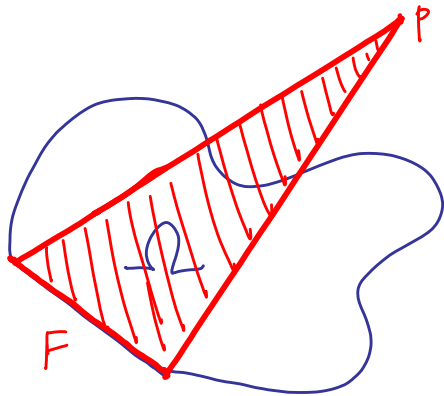
$$(a, \frac{1}{a}, 1) \longleftrightarrow [at, \frac{t}{a}, t]$$



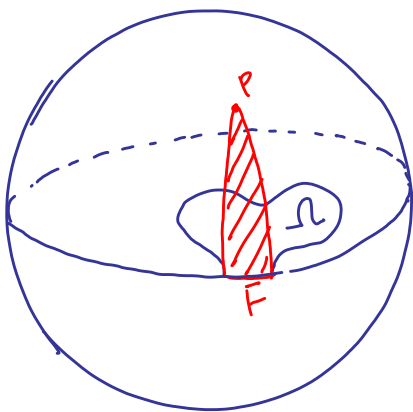
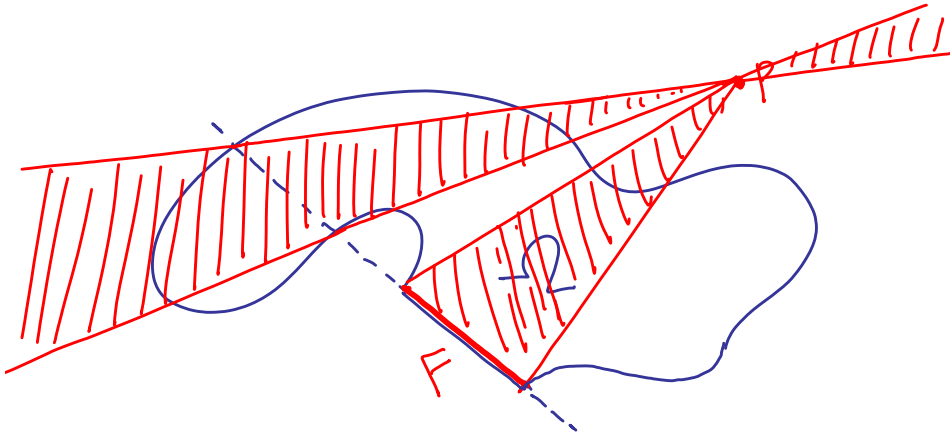
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$(X+Y)(X-Y) = Z^2$$

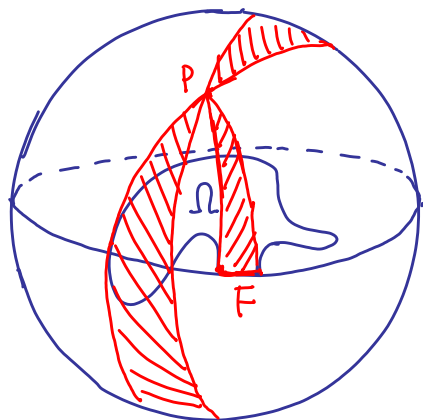




$\exists g_n \in PGL(3, \mathbb{R})$ s.t.
 $g_n(\Omega) \rightarrow C(F)$



$g_n(\Omega) \rightarrow C(F)$



$g_n(\Omega) \not\rightarrow C(F)$