A characterization of cones in the projective space

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Cones in the projective space

**Definition 1.** $\Omega :$ a domain in $\mathbb{RP}^n$

$B :$ a domain of a hyperplane $H$ of $\mathbb{RP}^n$

- $C(B) :$ a cone over $B$
  - a cone with the infinite boundary $\overline{B}$ in the affine space $\mathbb{A}^n = \mathbb{RP}^n \setminus H$,

- $\{b\} \vee B :$ a cone over $B$ with a cone point $b$.

(i) $C(B)$ is projectively equivalent to each component of $\pi^{-1}(B)$, where $\pi : \mathbb{R}^n \rightarrow H \simeq \mathbb{RP}^{n-1}$,

(ii) There are two cones over $B$ with a cone point $b$,

(iii) $C(B)$ is well-defined up to projective equivalence (not depending on the cone point)

(iv) $\{b\} \vee B = \{b\} + B$, if $B$ is properly convex.
Cones in the projective space - convex sums

- A properly convex domain $\Omega$ is called a *convex sum* of its faces $F_1$ and $F_2$, which will be denoted by

$$\Omega = F_1 + F_2,$$

if it is the interior of the convex hull of $\overline{F_1} \cup \overline{F_2}$ when we consider $\Omega$ as a bounded set in an affine space $\mathbb{A}^n$ in $\mathbb{RP}^n$, i.e., it is the union of all open line segments joining points in $F_1$ to points in $F_2$.

- Note that if the dimensions of $F_1$, $F_2$ and $\Omega$ are $k_1$, $k_2$ and $n$ respectively, then $n = k_1 + k_2 + 1$. 
Domains with flat boundary pieces

We say $\partial \Omega$ is \textit{locally flat} at $p$ if there is a hyperplane $H$ and an open ball $B_p$ centered at $p$ such that $\Omega \cap B_p$ is an open half ball with $H \cap B_p \subset \partial \Omega$.

**Definition 2.** Let $\Omega$ be a domain in $\mathbb{RP}^n$.

(i) $\Omega$ has a \textit{flat boundary piece} $P$ if $P^0$ is a connected open subset of hyperplane $H$ with $P^0 = P$ and $\partial \Omega$ is locally flat at each $p \in P^0$,

(ii) $\Omega$ has a \textit{strongly flat boundary piece} $P$ if $P$ is a flat boundary piece of $\Omega$ with $\langle P \rangle = H$ and there is an open neighborhood $U$ of $P$ such that $\Omega \cap U$ is contained in an open half space $H^+$ with boundary $H$. 
Quasi-homogeneous domains

**Definition 2.** $\Omega$ is a domain in $\mathbb{R}^n$ (or $\mathbb{RP}^n$).

(i) $\Omega$ is *quasi-homogeneous* if $\exists$ a compact subset $K \subset \Omega$ and a subgroup $G$ of Aut($\Omega$) such that $GK = \Omega$.

(ii) $\Omega$ is *divisible* if $\exists$ a cocompact discrete subgroup $H$ of Aut($\Omega$) acting on $\Omega$ properly.

(iii) $\Omega$ is a *homogeneous domain* if Aut($\Omega$) acts on $\Omega$ transitively.

If $M$ is a compact affine(projective, resp.) manifold and $D$ is a developing map from $\tilde{M}$ to $\mathbb{R}^n (\mathbb{RP}^n$, resp.), then

- $D(\tilde{M})$ is quasi-homogeneous.
- $D(\tilde{M})$ is divisible if $D$ is a diffeomorphism onto $D(\tilde{M})$. 
Previous results

- **Vinberg**(1963) classified all homogeneous convex cones algebraically.

- **Kuiper**(1953) classified 2-dimensional quasi-homogeneous convex domains while he was classifying convex compact projective surfaces.

- **Vey**(1970)
  Any quasi-homogeneous properly convex affine domain is a cone if it contains an open cone.

- **Benzécri**(1960)
  Any quasi-homogeneous properly convex projective domain with a face $F$ of codimension 1 is the convex sum of $F$ and a point in the boundary.

- **Benoist** (2000s) has been studying divisible convex domains and found many interesting examples.
A characterization of convex cones  

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Thm 1. \( \Omega \subset \mathbb{RP}^n \): properly convex domain

\( F : (n - 1) \)-dimensional face of \( \Omega \)

\( \text{Aut}_{\text{proj}}(\Omega)x \) accumulates to a point in \( F \), \( x \in \Omega \)

Then

\[ \Omega = \{ \xi \} + F, \quad \xi \in \partial \Omega. \]

Proof.

\[ \exists x \in \Omega, \{ g_i \} \subset \text{Aut}_{\text{proj}}(\Omega) \text{s.t.} \lim_{i \to \infty} g_i(x) = p \in F. \]

\[ \Rightarrow g = \lim_{i \to \infty} g_i, \text{Ran}(g) = \langle F \rangle, g(\Omega) = F \]

and \( \text{Ker}(g) = \{ z \} \) is an extreme point.

Case 1: \( z \notin \langle F \rangle \)

Case 2: \( z \in \langle F \rangle \)
Case 1: $z \notin \langle F \rangle$

\[
\lim_{i \to \infty} g_i(F) = g(F) = \overline{F}
\]

$\Rightarrow g(\{z\} \lor F) = F, g(\{z\} \lor (\langle F \rangle \setminus \overline{F})) = \langle F \rangle \setminus \overline{F}$

$\Rightarrow \Omega = \{z\} \lor F = \{z\} \dot{+} F$, one of two convex sums by connectedness of $\Omega$.

Case 2: $z \in \langle F \rangle$

$E = \{b \in \partial\Omega \mid bz \cap \Omega \neq \emptyset\}$

$\Rightarrow E$ is an $(n - 1)$-dimensional face of $\Omega$.

$\Rightarrow \{z\} \dot{+} E = \Omega$

$\Rightarrow \Omega = g_k(\Omega) = \{g_k(z)\} \dot{+} g_k(E) = \{g_k(z)\} \dot{+} F$

for some $k$, since $g_i(E)$ uniformly converges to $g(E) \subset F$. \hfill \Box
Coro 1. Let $\Omega$ be a convex domain in $\mathbb{R}^n$ and $F$ an $(n-1)$-dimensional face of $\Omega$. Suppose that there is a sequence $\{g_i\}$ of affine transformations which preserve $\Omega$ and a point $x$ in the interior of $\Omega$ such that $\{g_i(x)\}$ accumulates to an interior point of $F$. Then

$$\Omega = \mathbb{R}^+ \times F.$$
A characterization of cones

**Thm 2.** Let $\Omega$ be a domain with a flat boundary piece $P$ satisfying

(i) $P$ is a component of $\langle P \rangle \cap \Omega$,

(ii) $P$ has no complete line.

Then $\Omega = C(P^0)$ iff there is an accumulation point $p \in P^0$ under the action of $\text{Aut}(\Omega)$. 
Thm 3. Let $\Omega$ be a quasi-homogeneous affine domain with a flat boundary piece $P$ satisfying

(i) $P$ is a component of $\langle P \rangle \cap \Omega$,

(ii) $P$ has no complete line.

Then $\Omega = \mathbb{R}^+ \times P^0$, which is projectively equivalent to $C(P^0)$.

- In convex case,
  
  $\text{Theorem 1} \implies \text{Vey(1970), Benzécri(1960)}$.
  
  ($:\because \text{every point in any face of properly convex domain is an accumulation point.}$)

- A quasi-homogeneous domain is stable. ($\times$)

- $\exists g_n$ such that $\lim_{n \to \infty} g_n(\Omega) = C(P^0)$. ($\times$)

So we cannot apply Benzécri’s idea to non-convex case.

Proof. (i) Show that $\exists$ a sequence $g_n$ and $x \in \Omega$ such that $g_n(x)$ converges to a point in $P^0$.

(ii) Apply Theorem 2.
• $\mathcal{C}(n)$: the set of all convex bodies in $\mathbb{RP}^{n+1}$ (with the topology induced from the Hausdorff metric on the set of all closed subsets of $\mathbb{RP}^{n+1}$)

• $\mathcal{L}(n) = C(n)/ \text{PGL}(n + 1, \mathbb{R})$

• A convex body $C$ is stable
  $\iff \{[C]\} = \{[C]\}$ in $\mathcal{L}(n)$
  $\iff \{[C]\}$ is closed in $\mathcal{L}(n)$
  $\iff$ if $g_nC$ converges to a convex body $C'$, then $C = f(C')$ for some $f \in \text{PGL}(n + 1, \mathbb{R})$.

• The closure of a quasi-homogeneous properly convex domain is a stable convex body.
counter examples

The existence of an accumulation point in the flat boundary piece is not a sufficient condition for being a cone.

(i) $\Omega = \Omega_1 \cup \Omega_2$

$\Omega_1 = \{(x, y) \in \mathbb{R}^2 \mid x > 1\}$
$\Omega_2 = \{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1, y > 0\}$

$g_n = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$, $P = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, y = 0\}$.

- $P$ is a flat boundary piece of $\Omega$,
- $\lim_{n \to \infty} g_n(1/2, 1) = (1/2, 0) \in P^0$,
- $\Omega$ is not a cone.

(ii) $\Omega = \Omega_1 \cup \Omega_2$

$\Omega_1 = \{(x, y) \in \mathbb{R}^2 \mid x \leq 0, y > 0\}$
$\Omega_2 = \{(x, y) \in \mathbb{R}^2 \mid x > 0, 0 < y < 1/x\}$

$g_n = \begin{pmatrix} 2^n & 0 \\ 0 & \frac{1}{2^n} \end{pmatrix}$, $P = \{(x, y) \in \mathbb{R}^2 \mid y = 0\}$.

- $P$ is a flat boundary piece of $\Omega$,
- $\lim_{n \to \infty} g_n(0, y) = (0, 0) \in P^0$,
- $\Omega$ is not a cone.
Counter example (i)

\[ g_n = \left( \frac{1}{n}, \frac{1}{n^2} \right) \]

Counter example (ii)

\[ g_n = \left( \frac{2^n}{n}, \frac{1}{2^n} \right) \]
\[ \Omega = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y > 0\} \]

\( \Omega \) is not a cone in our sense.
But \( \Omega \) is a cone in a vector space \( \mathbb{R}^2 \).
P is a flat boundary piece of $\Omega$, but not strongly flat.
$\text{IRP}^3 \setminus H \cong \mathbb{E}^2$. 
Case 1.

\( \Omega_1, \Omega_2 \): convex sums of \( F \) and \{z\}.

\[ F + \{z\} \]
\[ C(F), \ F \cup \{z\} \]

\( \Omega_3, \Omega_4 : (\langle F \rangle \setminus \overline{F}) \cup \{z\} \).

\[ g(\Omega_1 \cup \Omega_2) = F \]
\[ g(\Omega_3 \cup \Omega_4) = \langle F \rangle \setminus \overline{F} \]
\[ g(\Omega) = F \]

\[ \Rightarrow \] \( \Omega \) is either \( \Omega_1 \) or \( \Omega_2 \).
Cone over a point

Cone over an interval

Cone over a triangle

Cone over a ball
Cone over a strictly convex domain
$x^2 + y^2 = 1$

$y = x^2$

$(a, \frac{1}{a}, 1) \leftrightarrow [at. \frac{t}{a}, t]$

$xy = z^2$

$(x, y, z) = \left( \frac{1}{2}, \frac{1}{2}, 0 \right)$

$(x+y)(x-y) = z^2$

$x^2, y^2, z^2$

$X = Y^2 + Z^2$
\[ g_n \in \operatorname{PGL}(3, \mathbb{R}) \text{ s.t. } g_n(\mathcal{H}) \rightarrow C(F) \]

\[ g_n(\mathcal{H}) \rightarrow C(F) \]

\[ g_n(\mathcal{H}) \rightarrow C(F) \]
A quasi-homogeneous domain is stable. (×)

\[ g_n = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{n} \end{pmatrix} \]

\[ g_n \Omega \rightarrow \]
Affine domains and projective domains

Affine domains can be considered as projective domains by the following equivariant embedding:

\[(i, \rho) : (\mathbb{E}^n, Aff(n)) \rightarrow (\mathbb{R}P^n, PGL(n + 1, \mathbb{R}))\]

\[i(x_1, \ldots, x_n) = [x_1, \ldots, x_n, 1]\]

\[\rho(A, a) = \begin{pmatrix} A & a \\ 0 & 1 \end{pmatrix}\]
Vey: Any quasi-homogeneous properly convex affine domain is a cone if it contains an open cone.
Benzécri: \( \Omega: \) quasi-homogeneous properly convex domain
\( F: \) face of \( \Omega \) with codim 1.

\[ \Rightarrow \Omega = F + \{b\} \quad \text{for some} \ b \in \partial \Omega \]

\[ \exists \ g \in \text{PGL}(3, \mathbb{R}) \ \text{st.} \]
\[ g^n(\Omega) \rightarrow F + \{p_3\} \]

\[ g = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \]
\[ g^n(\Omega) \rightarrow F \times \mathbb{R}^+ \quad (\text{in} \ \mathbb{R}^2) \]
\[ F + \{p_3\} \quad (\text{in} \ \mathbb{R}P^2) \]

By stability of quasi-homogeneous properly convex domains,
\( \Omega \) is projectively equivalent to \( F + \{p_3\} \).