Generators of $\mathrm{SL}(2,\mathbb{C})$-Character Varieties of Arbitrary Rank Free Groups

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Outline of Presentations

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- Skein modules and Sikora Graphs (or geometric motivation)

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- Algebraic Independence in the Moduli (or local coordinates)
- Magnus Trace Map (or limitations of locality)

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- Loosely speaking $\mathcal{S}(M, \mathbb{C}, q)$ is the space of n-valent ribbon graphs (isotopy classes) subject to certain relations satisfied by the "quantum invariant of the graph."
- He further shows $\mathcal{S}(M, \mathbb{C}, 1)$ is algebraically isomorphic to the SL$(n, \mathbb{C})$-character variety of $\pi_1(M)$.
- Explicit minimal generators and relations of such character varieties are only know in general for SL$(2, \mathbb{C})$ when $\pi_1(M)$ is free. We will discuss what is known in this case.

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- Let $\mathbb{C}[x_{ij}^k]/\Delta$ be the complex polynomial ring in $4r$ variables ($1 \leq k \leq r$ and $1 \leq i, j \leq 2$), where Δ is the ideal generated by the r irreducible polynomials

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- It is not hard to see that $\mathbb{C}[\mathfrak{R}] = \mathbb{C}[x^k_{ij}]/\Delta$.

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- We note that

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are simply called *generic matrices*.

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- The ring of invariants $\mathbb{C}[\mathfrak{R}]^{\mathfrak{G}}$ is a finitely generated domain, so

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- It is the variety whose coordinate ring is the ring of invariants; that is,

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Comment 0.1. *Some of these equivalences are not at all obvious, see Procesi and M. Artin for proofs.*

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Comment 0.3. *This theorem naturally generalizes to* $\mathrm{SL}(n,\mathbb{C})$.

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And in 1976 Procesi proved (in the context of $n \times n$ generic matrices)

Theorem 0.4 (Procesi). $\mathbb{C}[\mathfrak{Y}_r]$ *is generated by the invariants* $\mathrm{tr}(\mathbf{X}_{i_1}\mathbf{X}_{i_2}\cdots\mathbf{X}_{i_k})$*, where* $\mathbf{X}_j$ *are generic matrices.*

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Finding minimal generators amounts to finding all linear relations among generators of the same multidegree in the vector space

$$\mathbb{C}[\mathfrak{Y}_r]^+ / \left(\mathbb{C}[\mathfrak{Y}_r]^+\right)^2$$

where $\mathbb{C}[\mathfrak{Y}]^+$ is the ideal of positive terms.

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The Cayley-Hamilton equation gives

$$\mathbf{X}^2 - \mathrm{tr}(\mathbf{X})\mathbf{X} + \det(\mathbf{X})\mathbf{I} = 0.$$

And if we assume $\det(\mathbf{X}) = 1$, as is the case in $\mathbb{C}[\mathfrak{X}_r]$, we easily derive $\mathrm{tr}(\mathbf{X}^{-1}) = \mathrm{tr}(\mathbf{X})$ and $\mathrm{tr}(\mathbf{X}^2) = \mathrm{tr}(\mathbf{X})^2 - 2$.

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Comment 0.5. *Minimal generators for* $\mathbb{C}[\mathfrak{Y}_r]$ *were first worked out by Sibirskii in* 1968.

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However, the dimension of this variety is computed to be 2. Thus there can be no further relations.

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Theorem 0.6 (Lawton)**.** *Let $\mathcal{MG}(R)$ be the set of minimal generators of a ring R. Then*

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Moreover, let $\mathrm{N}_r(w,l)$ be the number of minimal generators of word length w in l letters choosen out of r possible. Then

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This generalizes to $\mathrm{SL}(n,\mathbb{C})$, just replace the $\mathbf{X}^2$ by $\mathbf{X}^n$.

Consequently, there are exactly r generators of type $\mathrm{tr}(\mathbf{X})$ in $\mathbb{C}[\mathfrak{X}_r]$ and none of type $\mathrm{tr}(\mathbf{X}^n)$, $n \neq 1$; and this is minimal among these generators.

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Since this accounts for the r reductions of minimal generators coming from projecting to $\mathbb{C}[\mathfrak{X}_r]$ there are no further losses of generators.

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So for the case, $\mathbb{C}[\mathfrak{Y}_2]$ we are left with the generators $\mathrm{tr}(\mathbf{X}_1), \mathrm{tr}(\mathbf{X}_2), \mathrm{tr}(\mathbf{X}_1^2), \mathrm{tr}(\mathbf{X}_2^2), \mathrm{tr}(\mathbf{X}_1, \mathbf{X}_2)$ since any other expression in two letters would result in a sub-expression with an exponent greater than one, which we just showed was impossible.

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Consequently, there are $\binom{r}{2}$ generators of type $\mathrm{tr}(\mathbf{X}\mathbf{Y})$ in $\mathbb{C}[\mathfrak{X}_r]$.

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Replacing $\mathbf{X}$ with $\mathbf{X}+\mathbf{Y}$ in the Cayley-Hamilton equation gives

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Simplifying this expression yields

$$\mathbf{XY}+\mathbf{YX} = \mathrm{tr}(\mathbf{X})\mathbf{Y} + \mathrm{tr}(\mathbf{Y})\mathbf{X} - \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y})\mathbf{I} + \mathrm{tr}(\mathbf{XY})\mathbf{I}.$$

Second step, but an important step...

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Multiplying on the right by $\mathbf{Z}$ we get the expression

$$\begin{aligned}\operatorname{tr}(\mathbf{XYZ}) + \operatorname{tr}(\mathbf{YXZ}) = & \operatorname{tr}(\mathbf{X})\operatorname{tr}(\mathbf{YZ}) + \operatorname{tr}(\mathbf{Y})\operatorname{tr}(\mathbf{XZ}) \\ & - \operatorname{tr}(\mathbf{X})\operatorname{tr}(\mathbf{Y})\operatorname{tr}(\mathbf{Z}) + \operatorname{tr}(\mathbf{XY})\operatorname{tr}(\mathbf{Z}).\end{aligned}$$

Last step...

Now taking this relation and substituting $\mathbf{Z} \mapsto \mathbf{ZW}$ gives a relation for $\operatorname{tr}(\mathbf{XYZW}) + \operatorname{tr}(\mathbf{YXZW})$.

And substituting $\mathbf{Y} \mapsto \mathbf{WY}$ gives $\operatorname{tr}(\mathbf{XWYZ}) + \operatorname{tr}(\mathbf{WYXZ})$.

Sending $\mathbf{X} \mapsto \mathbf{XW}$ gives $\operatorname{tr}(\mathbf{XWYZ}) + \operatorname{tr}(\mathbf{YXWZ})$; and $\mathbf{Z} \mapsto \mathbf{WZ}$ gives $\operatorname{tr}(\mathbf{XYWZ}) + \operatorname{tr}(\mathbf{YXWZ})$. Subtracting, adding, and subtracting these four relations gives $\operatorname{tr}(\mathbf{XYZW}) - \operatorname{tr}(\mathbf{YXWZ})$.

Fundamental Relation

However, sending $\mathbf{X} \mapsto \mathbf{W} \mapsto \mathbf{Y} \mapsto \mathbf{Z} \mapsto \mathbf{X}$ in the first expression gives $\mathrm{tr}(\mathbf{XYZW}) + \mathrm{tr}(\mathbf{XYWZ})$. This adds to our sum to give

$$\begin{aligned}
2\mathrm{tr}(\mathbf{XYZW}) =& \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{Z})\mathrm{tr}(\mathbf{W}) + \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{YZW}) \\
&+ \mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{XZW}) + \mathrm{tr}(\mathbf{Z})\mathrm{tr}(\mathbf{XYW}) + \mathrm{tr}(\mathbf{W})\mathrm{tr}(\mathbf{XYZ}) \\
&- \mathrm{tr}(\mathbf{XZ})\mathrm{tr}(\mathbf{YW}) + \mathrm{tr}(\mathbf{XW})\mathrm{tr}(\mathbf{YZ}) + \mathrm{tr}(\mathbf{XY})\mathrm{tr}(\mathbf{ZW}) \\
&- \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{ZW}) - \mathrm{tr}(\mathbf{X})\mathrm{tr}(\mathbf{W})\mathrm{tr}(\mathbf{YZ}) \\
&- \mathrm{tr}(\mathbf{Y})\mathrm{tr}(\mathbf{Z})\mathrm{tr}(\mathbf{XW}) - \mathrm{tr}(\mathbf{Z})\mathrm{tr}(\mathbf{W})\mathrm{tr}(\mathbf{XY})
\end{aligned}$$

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Comment 0.8. *It is clear that these functions are contained in the ring of invariants, but it is not obvious that these are all of them (this is what Procesi proved). Even more surprising is that the maximum word length is independent of the rank* r *(this is what we just showed!).*

Example $r = 3$

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If $\mathrm{tr}(\mathbf{XYZ})$ was allowed to be eliminated, we would conclude that $\mathfrak{X}_3$ was affine $\mathbb{C}^6$.

However, it is not hard to show there exists two representations which agree on the six generators of word length two or less but differ at $\mathrm{tr}(\mathbf{XYZ})$.

For instance,

$$\mathbf{X} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathbf{Y} = \begin{pmatrix} 0 & 2 \\ -1/2 & 0 \end{pmatrix}, \text{and } \mathbf{Z} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \text{ or}$$

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Comment 0.9. *One can further show there exists a product relation for* $\mathrm{tr}(\mathbf{XYZ})\mathrm{tr}(\mathbf{YXZ})$*. Together with the sum relation, we conclude that* $\mathfrak{X}_3$ *is a hypersurface and the generator of the ideal is a quadratic polynomial in* $\mathrm{tr}(\mathbf{XYZ})$ *over the other 6 generators.*

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Comment 0.10. *Sibirskii and Formanek were the first to work out the ideals in* $\mathbb{C}[\mathfrak{Y}_2]$ *and* $\mathbb{C}[\mathfrak{Y}_3]$; *Fricke and Vogt were the first to work out the case* $\mathbb{C}[\mathfrak{X}_2]$.

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Moreover, the minimal generators are:

$\mathcal{G}_1 = \{\mathrm{tr}(\mathbf{X}_1), ..., \mathrm{tr}(\mathbf{X}_r)\}$ of order r.

$\mathcal{G}_2 = \{\mathrm{tr}(\mathbf{X}_i\mathbf{X}_j) \mid 1 \leq i, j \leq r$ and $i \neq j\}$ of order $\binom{r}{2} = \frac{r(r-1)}{2}$.

$\mathcal{G}_3 = \{\mathrm{tr}(\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k) \mid 1 \leq i < j < k \leq r\}$ of order $\binom{r}{3} = \frac{r(r-1)(r-2)}{3}$.

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We conclude the following global geometric result:

Corollary 0.12. *The smallest affine embedding* $\mathfrak{X}_r \longrightarrow \mathbb{C}^{N_r}$ *is when* $N_r = \frac{r(r^2+5)}{6}$.

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The explicit computation of the minimal generators provides us with a specific embedding, and so they may be thought of as *coordinate functions.*

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We wish to determine a subset of the coordinate functions (minimal generators) which are local coordinates; that is, generate a full dimensional tangent space.

Such a set cannot have any relations among themselves alone; that is, they are algebraically independent.

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Theorem 0.13. *The following subsets of minimal generators are together algebraically independent:*

$\{\mathrm{tr}(\mathbf{X}_i) \mid 1 \leq i \leq r\}$ *of order* r

$\{\mathrm{tr}(\mathbf{X}_1\mathbf{X}_i) \mid 2 \leq i \leq r\}$ *of order* $r-1$

$\{\mathrm{tr}(\mathbf{X}_2\mathbf{X}_i) \mid 3 \leq i \leq r\}$ *of order* $r-2$*.*

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Certianly there are $3r-3$ of these generators, which is equal to the Krull dimension of $\mathfrak{X}_r$. Consequently, if they are independent they are maximally so.

Sketch of Proof

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- We prove this by induction. For $r = 1, 2, 3$ this has already been established.
- For $r \geq 4$ we calculate the Jacobian matrix of these $3r - 3$ functions in the $3r - 3$ independent variables:
  (from $\mathbf{X}_1$) x^1_{11}
  (from $\mathbf{X}_2$) x^2_{11}, x^2_{22},
  (from $\mathbf{X}_k$) $x^k_{11}, x^k_{12}, x^k_{22}$

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- For $r \geq 4$ we calculate the Jacobian matrix of these $3r - 3$ functions in the $3r - 3$ independent variables:
  (from $\mathbf{X}_1$) x_{11}^1
  (from $\mathbf{X}_2$) $x_{11}^2, x_{22}^2,$
  (from $\mathbf{X}_k$) $x_{11}^k, x_{12}^k, x_{22}^k$
- Putting $\text{tr}(\mathbf{X}_r), \text{tr}(\mathbf{X}_1\mathbf{X}_r), \text{tr}(\mathbf{X}_2\mathbf{X}_r)$ in the last 3 rows we get a block diagonal matrix. By induction we must show these three traces are independent in the variables from $\mathbf{X}_r$.

We first show that the matrix entries we have chosen are independent:

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Then conjugating by $\begin{pmatrix} 1/\sqrt{x^2_{21}} & 0 \\ 0 & \sqrt{x^2_{21}} \end{pmatrix}$ allows us to assume that $\mathbf{X}_2 = \begin{pmatrix} x^2_{11} & x^2_{11}x^2_{22} - 1 \\ 1 & x^2_{22} \end{pmatrix}$.

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Lastly, using $\det(\mathbf{X}_k) = 1$ allows us to solve for x_{21}^k in the other generic matrices.

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Lastly, using $\det(\mathbf{X}_k) = 1$ allows us to solve for x^k_{21} in the other generic matrices.

Since this leaves us with only $3r - 3$ elements, they must be independent since the Krull dimension is $3r - 3$ also.

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If there was a relation the determinant would be identically zero and so any non-zero evaluation shows independence. □

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In the cases, $r = 1, 2, 3$ the natural map $\mathfrak{X}_r \rightarrow \mathbb{C}^{3r-3}$ is surjective and so there is always a slice.

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In the cases, $r = 1, 2, 3$ the natural map $\mathfrak{X}_r \to \mathbb{C}^{3r-3}$ is surjective and so there is always a slice.

Unfortunately, this is not always the case:

Theorem 0.14 (Florentino, 2007)**.** *$\mathfrak{X}_r \longrightarrow \mathbb{C}^{3r-3}$ is only surjective in the cases $r = 1, 2, 3$; but in general the image omits only a subset of a codimension 1 subspace.*