Central Functions and $\mathrm{SL}(2,\mathbb{C})$-Character Varieties

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Outline of Presentation

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- Review and Preliminaries

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- Decomposition Theorem

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- Central Functions

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- Examples

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- We proved yesterday that its coordinate ring $\mathbb{C}[\mathfrak{X}_r]$ is generated by the $\frac{r(r^2+5)}{6}$ minimal generators:
  $\mathcal{G}_1 = \{\mathrm{tr}(\mathbf{X}_1), \ldots, \mathrm{tr}(\mathbf{X}_r)\}$ of order r.
  $\mathcal{G}_2 = \{\mathrm{tr}(\mathbf{X}_i\mathbf{X}_j) \mid 1 \leq i, j \leq r \text{ and } i \neq j\}$ of order $\frac{r(r-1)}{2}$.
  $\mathcal{G}_3 = \{\mathrm{tr}(\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k) \mid 1 \leq i < j < k \leq r\}$ of order $\frac{r(r-1)(r-2)}{3}$.

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  $\mathcal{G}_3 = \{\mathrm{tr}(\mathbf{X}_i\mathbf{X}_j\mathbf{X}_k) \mid 1 \leq i < j < k \leq r\}$ of order $\frac{r(r-1)(r-2)}{3}$.
- It is our purpose to show there is another set of functions that generate the coordinate ring (not minimally), but lend themselves to a graphical interpretation; and have a very interesting and non-obvious ring structure.

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Let $V_0 = \mathbb{C} = V_0^*$ be the trivial representation of $\mathrm{SL}(2,\mathbb{C})$ and denote the standard basis for $\mathbb{C}^2$ by $e_1 = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $e_2 = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, and the dual basis by its transpose: e_1^* and e_2^*.

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Then the standard representation and its dual are

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Proposition 0.1. *The symmetric powers of the standard representation of* $\mathrm{SL}(2,\mathbb{C})$ *are all irreducible representations and moreover they comprise a complete list.*

Explicit terms for V_n

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Denote the projection of $v_1 \otimes v_2 \otimes \cdots \otimes v_n \in V^{\otimes n}$ to V_n by $v_1 \circ v_2 \circ \cdots \circ v_n$. There exist bases for V_n and V_n^*, given by the elements

$$\mathsf{n}_{n-k} = e_1^{n-k} e_2^k = \underbrace{e_1 \circ e_1 \circ \cdots \circ e_1}_{n-k} \circ \underbrace{e_2 \circ e_2 \circ \cdots \circ e_2}_{k} \quad \text{and}$$

$$\mathsf{n}_{n-k}^* = (e_1^*)^{n-k} (e_2^*)^k = \underbrace{e_1^* \circ e_1^* \circ \cdots \circ e_1^*}_{n-k} \circ \underbrace{e_2^* \circ e_2^* \circ \cdots \circ e_2^*}_{k},$$

respectively, where $0 \leq k \leq n$.

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$$\begin{aligned} g \cdot \mathsf{n}_{n-k} &= (g_{11}e_1 + g_{21}e_2)^{n-k}(g_{12}e_1 + g_{22}e_2)^k \\ &= \sum_{\substack{0 \le j \le n-k \\ 0 \le i \le k}} \binom{n-k}{j}\binom{k}{i} \left(g_{11}^{n-k-j} g_{12}^{k-i} g_{21}^{j} g_{22}^{i} \right) \mathsf{n}_{n-(i+j)}. \end{aligned}$$

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For the dual, $\mathrm{SL}(2,\mathbb{C})$ acts on V_n^* in the usual way:

$$(g \cdot \mathsf{n}_{n-k}^*)(v) = \mathsf{n}_{n-k}^*(g^{-1} \cdot v) \text{ for } v \in V_n.$$

The "dual" pairing between V_n and V_n^* is given by

$$\mathsf{n}_{n-k}^*(v_1 \circ v_2 \circ \cdots \circ v_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} (\mathsf{n}_{n-k})^*(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}),$$

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In particular,

$$\mathsf{n}_{n-k}^*(\mathsf{n}_{n-l}) = \frac{(n-k)!k!}{n!}\delta_{kl} = \delta_{kl} \Big/ \binom{n}{k}.$$

The tensor product $V_a \otimes V_b$, where $a, b \in \mathbb{N}$, is also a representation of $\mathrm{SL}(2, \mathbb{C})$ and decomposes into irreducible representations as follows:

Proposition 0.2 (Clebsch-Gordan formula)**.**

$$V_a \otimes V_b \approx \bigoplus_{j=0}^{\min(a,b)} V_{a+b-2j}.$$

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We remind ourselves of Schur's Lemma for later use:

Proposition 0.3 (Schur's Lemma)**.** *Let G be a group, V and W irreducible representations of G, and $f \in \mathrm{Hom}_G(V, W)$ with $f \neq 0$. If $V \approx W$, then $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 1$; and if $V \not\approx W$, then $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W) = 0$.*

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Theorem 0.4 (Decomposition). *There is a* $\mathrm{SL}(2,\mathbb{C})$*-module isomorphism*

$$\mathbb{C}[\mathrm{SL}(2,\mathbb{C})] \approx \sum_{k\in\mathbb{N}} V_k^* \otimes V_k \approx \sum_{k\in\mathbb{N}} \mathrm{End}(V_k).$$

The isomorphism is given by defining

$$\Upsilon : \sum_{n \geq 0} V_n^* \otimes V_n \longrightarrow \mathbb{C}[\mathrm{SL}(2, \mathbb{C})]$$

by linear extension of the mapping

$$\mathsf{n}_{n-k}^* \otimes \mathsf{n}_{n-l} \mapsto \mathsf{n}_{n-k}^*(\mathbf{X} \cdot \mathsf{n}_{n-l}),$$

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It particular,

$$\begin{aligned} \mathsf{n}_{n-k}^*(\mathbf{X} \cdot \mathsf{n}_{n-l}) &= \mathsf{n}_{n-k}^* \left((x_{11}e_1 + x_{21}e_2)^{n-l} (x_{12}e_1 + x_{22}e_2)^l \right) \\ &= \sum_{\substack{i+j=k \\ 0 \leq i \leq n-l \\ 0 \leq j \leq l}} \binom{n}{k}^{-1} \binom{n-l}{i} \binom{l}{j} x_{11}^{n-l-i} x_{12}^{l-j} x_{21}^{i} x_{22}^{j}. \end{aligned}$$

Applying the decompostion,

$$
\begin{aligned}
\mathbb{C}[\mathrm{SL}(2,\mathbb{C})^{\times r}] &\approx \mathbb{C}[\mathrm{SL}(2,\mathbb{C})]^{\otimes r} \\
&\approx \bigotimes_{1\le k\le r} \left(\sum_{i_k\in\mathbb{N}} V_{i_k}^* \otimes V_{i_k} \right) \\
&\approx \sum_{(i_1,\ldots,i_r)\in\mathbb{N}^r} V_{i_1}^* \otimes V_{i_1} \otimes \cdots \otimes V_{i_r}^* \otimes V_{i_r} \\
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Let $\vec{i} = (i_1, i_2, ..., i_r) \in \mathbb{N}^r$, and let $|\vec{i}| = i_1 + \cdots + i_r$.

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Definition 0.5. *We say that $\vec{j} = (j_1, ..., j_{r-1}) \in \mathbb{N}^{r-1}$ is $\vec{i}$-admissible (and denote it by $\vec{j} \in \lceil\vec{i}\rfloor$) if and only if for all $1 \leq k \leq r-1$ we have*

$$0 \leq j_k \leq \min(i_1 + \cdots + i_k - 2(j_1 + \cdots + j_{k-1}), i_{k+1}).$$

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&\approx \sum_{(i_1,\ldots,i_r)\in\mathbb{N}^r} V_{i_1}^* \otimes \cdots \otimes V_{i_r}^* \otimes V_{i_1} \otimes \cdots \otimes V_{i_r} \\
&\approx \sum_{\vec{i}\in\mathbb{N}^r} \left(\sum_{\vec{j}\in\lceil\vec{i}\rfloor} V^*_{\left(|\vec{i}|-2|\vec{j}|\right)} \right) \otimes \left(\sum_{\vec{k}\in\lceil\vec{i}\rfloor} V_{\left(|\vec{i}|-2|\vec{k}|\right)} \right) \\
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\end{aligned}$$

Since the above maps are $\mathrm{SL}(2,\mathbb{C})$-equivariant, $\mathbb{C}[\mathfrak{X}_r] =$

$$\mathbb{C}[\mathrm{SL}(2,\mathbb{C})^{\times r}]^{\mathrm{SL}(2,\mathbb{C})} \approx \sum_{\vec{i}\in\mathbb{N}^r} \sum_{\vec{j},\vec{k}\in\lceil\vec{i}\rfloor} \left(V^*_{\left(|\vec{i}|-2|\vec{j}|\right)} \otimes V_{\left(|\vec{i}|-2|\vec{k}|\right)} \right)^{\mathrm{SL}(2,\mathbb{C})}$$

By Schur's Lemma,

$$\dim_{\mathbb{C}} \left(V^*_{(|\vec{i}|-2|\vec{j}|)} \otimes V_{(|\vec{i}|-2|\vec{k}|)} \right)^{\mathrm{SL}(2,\mathbb{C})} = \begin{cases} 1 & \text{if } |\vec{k}| = |\vec{j}| \\ 0 & \text{if } |\vec{j}| \neq |\vec{k}| \end{cases}$$

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Therefore, $\mathbb{C}[\mathfrak{X}_r]=$

$$\mathbb{C}[\mathrm{SL}(2,\mathbb{C})^{\times r}]^{\mathrm{SL}(2,\mathbb{C})}\approx\sum_{\vec{i}\in\mathbb{N}^r}\sum_{\substack{\vec{j},\vec{k}\in\lceil\vec{i}\rfloor\\ |\vec{k}|=|\vec{j}|}}\mathrm{End}\left(V_{(|\vec{i}|-2|\vec{j}|)}\right)^{\mathrm{SL}(2,\mathbb{C})}.$$

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Definition 0.6. *Given the above isomorphism, for each triple* $\vec{i},\vec{j},\vec{k}$ *such that* $\vec{i}\in\mathbb{N}^r$, $\vec{j},\vec{k}\in\lceil\vec{i}\rfloor$, *and* $|\vec{j}|=|\vec{k}|$, *there exists a class function* $\chi_{\vec{i}}^{\vec{j},\vec{k}}\in\mathbb{C}[\mathfrak{X}_r]$ *which corresponds to a generating homothety (unique up to scalar) in* $\mathrm{End}(V_{(|\vec{i}|-2|\vec{j}|)})^{\mathrm{SL}(2,\mathbb{C})}$. *We refer to the functions* $\chi_{\vec{i}}^{\vec{j},\vec{k}}$ *as* central functions.

Denote by $\mathbb{C}\,\mathcal{X}_{\vec{i}}^{\vec{j},\vec{k}} \subset \mathbb{C}[\mathfrak{X}_r]$ the linear span over $\mathbb{C}$ of $\mathcal{X}_{\vec{i}}^{\vec{j},\vec{k}}$.

Denote by $\mathbb{C}\,\chi_{\vec{i}}^{\vec{j},\vec{k}} \subset \mathbb{C}[\mathfrak{X}_r]$ the linear span over $\mathbb{C}$ of $\chi_{\vec{i}}^{\vec{j},\vec{k}}$.

In these terms,

$$\mathbb{C}[\mathfrak{X}_r] \approx \sum_{\vec{i}\in\mathbb{N}^r} \sum_{\substack{\vec{j},\vec{k}\in\lceil\vec{i}\rfloor \\ |\vec{k}|=|\vec{j}|}} \mathbb{C}\,\chi_{\vec{i}}^{\vec{j},\vec{k}}.$$

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And so there is a ring structure. However, the multiplicative structure in terms of this basis is very complex and not at all obvious.

We note that $\vec{i}$ has r entries, $\vec{k}$ and $\vec{j}$ have $r-1$ and the index relation $|\vec{j}| = |\vec{k}|$ shows that each central function is in terms of exactly $3r-3$ indices, the Krull dimension of the variety.

With respect to the Clebsch-Gordan injection

$$\iota_{\vec{k}}^{\vec{i}} : V_{\left(|\vec{i}|-2|\vec{k}|\right)} \hookrightarrow V_{i_1} \otimes \cdots \otimes V_{i_r},$$

let

$$\mathrm{M}\left(\vec{i},\vec{j},\vec{k}\right) = \left(\iota_{\vec{j}}^{\vec{i}}(\mathsf{c}_s^*)\left((\mathbf{X}_1,...,\mathbf{X}_r)\cdot\iota_{\vec{k}}^{\vec{i}}(\mathsf{d}_t)\right)\right)_{st},$$

the $\left(|\vec{i}|-2|\vec{k}|+1\right)\times\left(|\vec{i}|-2|\vec{k}|+1\right)$ matrix with noted s,t entries; and where $\{\mathsf{c}_s^*\}$ is a basis for $V^*_{\left(|\vec{i}|-2|\vec{j}|\right)}$ and $\{\mathsf{d}_t\}$ is a basis for $V_{\left(|\vec{i}|-2|\vec{k}|\right)}$ (assuming $|\vec{j}|=|\vec{k}|$).

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Then,

$$\chi_{\vec{i}}^{\vec{j},\vec{k}}(\mathbf{X}_1,\ldots,\mathbf{X}_r) = \mathrm{tr}(\mathrm{M}(\vec{i},\vec{j},\vec{k})).$$

The functions $\chi_{\vec{i}}^{\vec{j},\vec{k}}$ take a natural diagrammatic form. This will be discussed in Elisha Peterson's talk. We refer to these diagrams as *Peterson Graphs.*

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1. The Clebsch-Gordon injections and its dual are intertwiners determined by the heighest weight action on the indices of a tensor $v_1 \otimes \cdots \otimes v_n$. Hence, this action is a signed sum of permutations.

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1. The Clebsch-Gordon injections and its dual are intertwiners determined by the heighest weight action on the indices of a tensor $v_1 \otimes \cdots \otimes v_n$. Hence, this action is a signed sum of permutations.

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3. We use cups and caps to represent the determinant and codeterminant maps, necessary for the injections (and their duals) $\mathbb{C} \hookrightarrow V_{i_1} \otimes \cdots \otimes V_{i_r}$.

4. We can then represent $\chi_{\vec{i}}^{\vec{j},\vec{k}} \in \mathrm{End}(V_{i_1} \otimes \cdots \otimes V_{i_r})^{\mathrm{SL}(2,\mathbb{C})}$ by a union of cups, braids, and caps; that is, a graph.

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Since they form a basis for the coordinate ring, and we know that the algebraic coordinates discussed yesterday correspond directly to Sikora graphs, we know that Peterson Graphs and Sikora Graphs can be written in terms of each other.

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Hence

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Note that the ring structure is NOT that of $\mathbb{C}[\mathrm{tr}(\mathbf{X})]$.

However, the product formula and the initial calculations of $\mathcal{X}_0$ and $\mathcal{X}_1$ may be used to again show: $\mathbb{C}[\mathrm{SL}(2,\mathbb{C})]^{\mathrm{SL}(2,\mathbb{C})} \approx \mathbb{C}[t]$.

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Suppose $a \geq 2$ and $\mathcal{X}_b$ is in the image for all $b < a$. Then $\mathcal{X}_1 \mathcal{X}_{a-1} = \mathcal{X}_a + \mathcal{X}_{a-2}$. Thus, by induction, the map is surjective.

Example $r = 2$

Recall the decomposition

$$\mathbb{C}[\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})]^{\mathrm{SL}(2,\mathbb{C})} \approx \sum_{\substack{a,b\in\mathbb{N} \\ c\in\lceil a,b\rfloor}} \mathbb{C}\, \chi_c^{a,b},$$

where $\chi_c^{a,b}$ corresponds to the image of

$$\sum_{k=0}^{c} \mathsf{c}_k (\mathsf{c}_k)^T \mapsto \sum_{k=0}^{c} \binom{c}{k} \mathsf{c}_k^* \otimes \mathsf{c}_k$$

under the injection $V_c^* \otimes V_c \hookrightarrow V_a^* \otimes V_b^* \otimes V_a \otimes V_b$.

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under the injection $V_c^* \otimes V_c \hookrightarrow V_a^* \otimes V_b^* \otimes V_a \otimes V_b$.

This inclusion is determined by the Clebsch-Gordan injection $\iota : V_c \hookrightarrow V_a \otimes V_b$. Hence, an explicit formula for ι provides a means to compute $\chi_c^{a,b}$ directly.

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For $k = 1, 2$, let $\mathbf{X}_k = [x^k_{ij}]$ be 2×2 generic matrices, and let

$$x = \operatorname{tr}(\mathbf{X}_1) = x^1_{11} + x^1_{22},$$

$$y = \operatorname{tr}(\mathbf{X}_2) = x^2_{11} + x^2_{22},$$

$$z = \operatorname{tr}(\mathbf{X}_1\mathbf{X}_2^{-1}) = (x^1_{11}x^2_{22} + x^1_{22}x^2_{11}) - (x^1_{12}x^2_{21} + x^1_{21}x^2_{12}).$$

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The map $\cup : V_0 \hookrightarrow V_1 \otimes V_1$ given by

$$\mathsf{c}_0 \mapsto \mathsf{a}_0 \otimes \mathsf{b}_1 - \mathsf{a}_1 \otimes \mathsf{b}_0$$

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More generally, the injection $V_0 \hookrightarrow V_a \otimes V_a$ is given by

$$\cup^a : \mathsf{c}_0 \longmapsto \sum_{m=0}^{a} (-1)^m \binom{a}{m} \mathsf{a}_{a-m} \otimes \mathsf{b}_m.$$

Hence, $\chi_0^{0,0} = 1$ and $\chi_0^{1,1}$ may be computed by:

$$\begin{aligned}
\chi_0^{1,1} &\mapsto \mathsf{c}_0^* \otimes \mathsf{c}_0 \\
&\mapsto (\mathsf{a}_0^* \otimes \mathsf{b}_1^* - \mathsf{a}_1^* \otimes \mathsf{b}_0^*) \otimes (\mathsf{a}_0 \otimes \mathsf{b}_1 - \mathsf{a}_1 \otimes \mathsf{b}_0) \\
&\mapsto (\mathsf{a}_0^* \otimes \mathsf{a}_0) \otimes (\mathsf{b}_1^* \otimes \mathsf{b}_1) - (\mathsf{a}_1^* \otimes \mathsf{a}_0) \otimes (\mathsf{b}_0^* \otimes \mathsf{b}_1) \\
&\qquad - (\mathsf{a}_0^* \otimes \mathsf{a}_1) \otimes (\mathsf{b}_1^* \otimes \mathsf{b}_0) + (\mathsf{a}_1^* \otimes \mathsf{a}_1) \otimes (\mathsf{b}_0^* \otimes \mathsf{b}_0) \\
&\mapsto x_{11}^1 \otimes x_{22}^2 - x_{12}^1 \otimes x_{21}^2 - x_{21}^1 \otimes x_{12}^2 + x_{22}^1 \otimes x_{11}^2 \\
&\mapsto (x_{11}^1 x_{22}^2 + x_{22}^1 x_{11}^2) - (x_{12}^1 x_{21}^2 + x_{21}^1 x_{12}^2) = z.
\end{aligned}$$

The representation V_c may be identified with a subset of $V^{\otimes c}$ via the equivariant maps

$$V_c \underset{\mathsf{Proj}}{\overset{\mathsf{Sym}}{\rightleftarrows}} V^{\otimes c}$$

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Thus, when $c = a + b$, ι is given by the commutative diagram

$$\begin{array}{ccc} V^{\otimes c} & = & V^{\otimes a} \otimes V^{\otimes b} \\ {\scriptstyle \mathsf{Sym}}\uparrow & \circlearrowright & \downarrow{\scriptstyle \mathsf{Proj}\otimes\mathsf{Proj}} \\ V_c & \xrightarrow[\iota]{} & V_a \otimes V_b. \end{array}$$

In particular,

$$\binom{c}{k}\mathsf{c}_k \stackrel{\iota}{\longmapsto} \sum_{\substack{0\le i\le a\\ 0\le j\le b\\ i+j=k}} \binom{a}{i}\mathsf{a}_i \otimes \binom{b}{j}\mathsf{b}_j.$$

For example, consider $\mathcal{X}_1^{1,0}$. In this case, $\mathsf{c}_0 \mapsto \mathsf{a}_0 \otimes \mathsf{b}_0$ and $\mathsf{c}_1 \mapsto \mathsf{a}_1 \otimes \mathsf{b}_0$.

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Hence,

$$\begin{aligned}\mathcal{X}_1^{1,0} &\mapsto \mathsf{c}_0^* \otimes \mathsf{c}_0 + \mathsf{c}_1^* \otimes \mathsf{c}_1 \mapsto (\mathsf{a}_0^* \otimes \mathsf{a}_0) \otimes (\mathsf{b}_0^* \otimes \mathsf{b}_0) + (\mathsf{a}_1^* \otimes \mathsf{a}_1) \otimes (\mathsf{b}_0^* \otimes \mathsf{b}_0) \\ &\mapsto x_{11}^1 \otimes 1 + x_{22}^1 \otimes 1 \mapsto x_{11}^1 + x_{22}^1 = x.\end{aligned}$$

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$$\mapsto x_{11}^1 \otimes 1 + x_{22}^1 \otimes 1 \mapsto x_{11}^1 + x_{22}^1 = x.$$

A similar computation shows that $\mathcal{X}_1^{0,1} \mapsto y$.

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It follows that the mapping $\iota : V_c \to V_a \otimes V_b$ is explicitly given by:

$$\binom{c}{k}\mathsf{c}_k \longmapsto \sum_{\substack{0\le i\le\beta \\ 0\le j\le\alpha \\ 0\le m\le\gamma \\ i+j=k}} \binom{\beta}{i}\mathsf{a}_i \otimes \left[(-1)^m\binom{\gamma}{m}\mathsf{a}_{\gamma-m}\otimes\mathsf{b}_m\right] \otimes \binom{\alpha}{j}\mathsf{b}_j$$

$$\longmapsto \sum_{\substack{0\le i\le\beta \\ 0\le j\le\alpha \\ 0\le m\le\gamma \\ i+j=k}} (-1)^m\binom{\beta}{i}\binom{\alpha}{j}\binom{\gamma}{m}\mathsf{a}_{i+\gamma-m}\otimes\mathsf{b}_{j+m}.$$

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Theorem 0.7 (Peterson, 2006)**.** *Provided $a > 1$ and $c > 1$, we can write*

$$\chi_c^{a,b} = x \cdot \chi_{a-1}^{b,c-1} - \frac{(a+b-c)^2}{4a(a-1)}\chi_c^{a-2,b} - \frac{(-a+b+c)^2}{4c(c-1)}\chi_{c-2}^{a,b} - \frac{(a+b+c)^2(a-b+c-2)^2}{16a(a-1)c(c-1)}\chi_{c-2}^{a-2,b}.$$

The relation still holds for $a = 1$ or $c = 1$, provided we exclude the terms with $a - 1$ or $c - 1$ in the denominator.

Also, note that formulae for multiplication by y and z may be obtained by applying the following symmetry relation.

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Suppose a central function is expressed as a polynomial P in the variables $x = \mathrm{tr}(\mathbf{X}_1)$, $y = \mathrm{tr}(\mathbf{X}_2)$, and $z = \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1})$, so that $P_{a,b,c}(y, x, z) = \chi_c^{a,b}(\mathbf{X}_1, \mathbf{X}_2)$ for some admissible triple (a, b, c).

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Theorem 0.8 (Peterson,2006)**.** *For any permutation σ,*

$$P_{\sigma(a,b,c)}(y,x,z) = P_{a,b,c}(\sigma^{-1}(y,x,z)).$$

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$$\Gamma : \mathbb{C}[t_x, t_y, t_z] \longrightarrow \mathbb{C}[\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})]^{\mathrm{SL}(2,\mathbb{C})}$$

by $f(t_x, t_y, t_z) \mapsto f(\mathrm{tr}(\mathbf{X}_1), \mathrm{tr}(\mathbf{X}_2), \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1}))$.

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Suppose $f(\mathrm{tr}(\mathbf{X}_1), \mathrm{tr}(\mathbf{X}_2), \mathrm{tr}(\mathbf{X}_1\mathbf{X}_2^{-1})) = 0$ for all pairs $(\mathbf{X}_1, \mathbf{X}_2) \in \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$.

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Let $(\tau_x, \tau_y, \tau_z) \in \mathbb{C}^3$, $\epsilon_x = \begin{bmatrix} \tau_x & 1 \\ -1 & 0 \end{bmatrix}$, and $\eta_{y,z} = \begin{bmatrix} \tau_y & \frac{1}{\zeta} \\ -\zeta & 0 \end{bmatrix}$, where $\zeta + \zeta^{-1} = \tau_z$.

Then

$$(\tau_x, \tau_y, \tau_z) = (\mathrm{tr}(\epsilon_x), \mathrm{tr}(\eta_{y,z}), \mathrm{tr}(\epsilon_x \eta_{y,z}^{-1})).$$

Hence $f = 0$ on $\mathbb{C}^3$, $\mathrm{Ker}(\Gamma) = \{0\}$, and Γ is injective. This is the "Fricke slice" given by Goldman.

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It remains to show that Γ is surjective.

The decomposition theorem implies that the central functions form a basis for $\mathbb{C}[\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})]^{\mathrm{SL}(2,\mathbb{C})}$. Since $t_x \mapsto x, t_y \mapsto y$, and $t_z \mapsto z$, it suffices to show that every $\chi_c^{a,b}$ may be written as a polynomial in x, y, and z.

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For the base cases $\delta = 0, 1$ recall our earlier computations demonstrating

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For $\delta > 0$, we may inductively assume that all central functions with rank less than δ are in $\mathbb{C}[x,y,z]$. The admissibility conditions imply that at least two out of the triple (a,b,c) are positive. Without loss of generality, using the Symmetry Theorem, we may assume that a and c are positive.

In this case, the recursion formula gives $\chi_c^{a,b} =$

$$x \cdot \chi_{c-1}^{a-1,b} - \frac{(a+b-c)^2}{4a(a-1)}\chi_c^{a-2,b} - \frac{(-a+b+c)^2}{4c(c-1)}\chi_{c-2}^{a,b} - \frac{(a+b+c)^2(a-b+c-2)^2}{16a(a-1)c(c-1)}\chi_{c-2}^{a-2,b},$$

allows us to write $\chi_c^{a,b}$ in terms of central functions of lower rank, which by induction must be in $\mathbb{C}[x,y,z]$.

In this case, the recursion formula gives $\chi_c^{a,b} =$

$$x \cdot \chi_{c-1}^{a-1,b} - \frac{(a+b-c)^2}{4a(a-1)}\chi_c^{a-2,b} - \frac{(-a+b+c)^2}{4c(c-1)}\chi_{c-2}^{a,b} - \frac{(a+b+c)^2(a-b+c-2)^2}{16a(a-1)c(c-1)}\chi_{c-2}^{a-2,b},$$

allows us to write $\chi_c^{a,b}$ in terms of central functions of lower rank, which by induction must be in $\mathbb{C}[x,y,z]$.

Thus, $\chi_c^{a,b} \in \mathbb{C}[x,y,z]$, and we have established surjectivity.

Peterson also worked out the ring structure, but we do not go into it here, since it is quite complex and entirly graphical.

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For $r > 2$, this ring structure is not known.