Diagrammatic Central Functions

KAIST Geometric Topology Fair

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Outline

1. The Central Function Basis
   - Algebraic Approach
   - Diagrammatic Approach

2. Trace Diagrams and Representation Theory
   - Representations and Tensor Algebra
   - $SL(2, \mathbb{C})$ Trivalent Diagrams

3. Computation of Central Functions
   - Rank One
   - Rank Two
   - Rank Three

4. Questions for Exploration
   - Computing $SL(2, \mathbb{C})$ Central Functions
   - Generalizations
Basis for the Coordinate Ring I

Diagrams can be used to construct a basis for the trace ring.

Theorem

The polynomials $\chi_{c}^{a,b}$ comprise a basis for the coordinate ring of the $\text{SL}(2, \mathbb{C})$-character variety of the three-holed sphere.

Proof uses the unitary trick and the Peter-Weyl Theorem.

Expand symmetrizers and remove crossings to obtain a trace polynomial

$$\chi_{5,6}^{7}(\text{tr}(A), \text{tr}(B), \text{tr}(A\bar{B}))$$
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- Proof uses the unitary trick and the Peter-Weyl Theorem.
The Peter-Weyl Theorem

The Peter-Weyl Theorem provides a means to describe a basis of functions for a coordinate ring.

**Theorem (Corollary of Peter-Weyl)**

The coordinate ring $\mathbb{C}[G]$ for a reductive linear algebraic group $G$ decomposes:

$$\bigoplus_{\lambda \in \Lambda} V_{\lambda}^* \otimes V_{\lambda} \cong \mathbb{C}[G],$$

where $\Lambda$ is the set of irreducible representations (of the maximal compact subgroup $U \subset G$), and the isomorphism is given by

$$\nu^* \otimes w \mapsto (x \mapsto \nu^*(x \cdot w)).$$
Theorem (Central Function Decomposition)

The coordinate ring of the character variety may be decomposed

\[ \mathbb{C}[\mathcal{X}_r] \cong \bigoplus_{\tilde{\lambda} \in \mathcal{X}_r} \bigoplus_{\psi = \phi \in [\tilde{\lambda}]} \mathbb{C} \chi_{\tilde{\lambda}}^{\psi, \phi}, \]

where \( \psi = \phi \in [\tilde{\lambda}] \) indicates that \( V_\psi, V_\phi \hookrightarrow V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r} \), but may possibly be different injections.

Definition (Central Functions)

The Central Functions of the \( G \)-character variety \( \chi_r \) are the functions \( \chi_{\tilde{\lambda}}^{\psi, \phi} \) in the above decomposition.
Central Functions of the Character Variety

**Theorem (Central Function Decomposition)**

The coordinate ring of the character variety may be decomposed

\[ \mathbb{C}[\mathcal{X}_r] \cong \bigoplus_{\tilde{\lambda} \in \Lambda^r} \bigoplus_{\psi = \phi \in \tilde{\lambda}} \mathbb{C} \chi_{\tilde{\lambda}}^{\psi, \phi}, \]

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**Definition (Central Functions)**

The *Central Functions* of the \( G \)-character variety \( \chi_r \) are the functions \( \chi^{\psi, \phi}_{\tilde{\lambda}} \) in the above decomposition.
Proof of the Central Function Decomposition

Proof.

When a surface $\Sigma$ has fundamental group free of rank $r$, the isomorphism $\mathbb{C}[\text{Hom}(\pi, G)] \cong \mathbb{C}[G^r] \cong \mathbb{C}[G]^r$ and the previous result give:

$$\mathbb{C}[\mathcal{X}] \cong (\mathbb{C}[G]^r)^G \cong \left( \bigotimes_{\lambda \in \Lambda} V_{\lambda}^* \otimes V_{\lambda} \right)^G \cong \bigoplus_{(\lambda_1, \ldots, \lambda_r) \in \Lambda^r} \left( \bigotimes_{\lambda_1} (V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_r}^*) \otimes (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}) \right)^G.$$

Schur’s Lemma and $G$-invariance permit a reduction to the desired form:

$$\mathbb{C}[\mathcal{X}_r] \cong \bigoplus_{\tilde{\lambda} \in \Lambda^r} \bigoplus_{\psi = \phi \in [\tilde{\lambda}]} \mathbb{C} \chi_{\tilde{\lambda}}^{\psi, \phi}.$$
The Diagrammatic Basis

Steps to the Diagrammatic Representation.

1. Represent $\mathbb{C}\chi_\lambda$ diagrammatically.
2. Represent the injections $V_\psi, V_\phi \hookrightarrow V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$ diagrammatically.
3. Combine the injections, the $G^{\times r}$-action, and the trace property.
The Diagrammatic Basis

Steps to the Diagrammatic Representation.

1. Represent $C_{\chi \lambda}$ diagrammatically.

The isomorphism $V_\lambda^* \otimes V_\lambda \cong C_{\chi \lambda}$ is defined for a basis $\{v_i\}$ of $V_\lambda$ by

$$v^* \otimes w \mapsto \text{tr}(x \mapsto v^*(x \cdot w)) = \sum_i v_i^*(x \cdot v_i).$$

The corresponding diagram is $C_{\chi \lambda} = \bigotimes^\lambda$.

2. Represent the injections $V_\psi, V_\phi \hookrightarrow V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$ diagrammatically.

3. Combine the injections, the $G^{\times r}$-action, and the trace property.
Steps to the Diagrammatic Representation.

1. Represent $\mathbb{C} \chi_\lambda$ diagrammatically.
2. Represent the injections $V_\psi, V_\phi \leftrightarrow V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$ diagrammatically.
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The Diagrammatic Basis

Steps to the Diagrammatic Representation.

1. Represent $C_{\chi\lambda}$ diagrammatically.
2. Represent the injections $V_\psi, V_\phi \hookrightarrow V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$ diagrammatically.

Each such injection corresponds to a term in the decomposition of this tensor product into irreducible elements.

In this diagram, each node represents an injection $V_\alpha \hookrightarrow V_\beta \otimes V_\gamma$, and the tree gives a well-defined way to perform this decomposition.
The Diagrammatic Basis

Steps to the Diagrammatic Representation.

1. Represent $\mathbb{C} \chi_\lambda$ diagrammatically.
2. Represent the injections $V_\psi, V_\phi \leftarrow V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$ diagrammatically.
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Steps to the Diagrammatic Representation.

1. Represent $\mathbb{C} \chi_\lambda$ diagrammatically.
2. Represent the injections $V_\psi, V_\phi \hookrightarrow V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$ diagrammatically.
3. Combine the injections, the $G^{\times r}$-action, and the trace property.
A cut set as defined in the previous talk permits “opening” up a surface onto the plane.

Definition

A cut triangulation is an extension of a cut set which divides the surface into a set of triangles (with neighborhoods of vertices removed).

Cut triangulations provide canonical decompositions of

\[ ((V^*_{\lambda_1} \otimes \cdots \otimes V^*_{\lambda_r}) \otimes (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}))^G. \]
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\[
((V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_r}^*) \otimes (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}))^G.
\]
The Diagrammatic Basis Theorem

**Theorem**

Let $\Sigma$ be a compact surface with boundary. Given a cut triangulation extending a specified cut set, every $G$-admissible labelling of its dual 1-skeleton induces a trace diagram which is identified with a $G$-invariant function $\text{Hom}(\pi, G) \to \mathbb{C}$. Moreover, for every cut triangulation, the set of such diagrams is a basis for $\mathbb{C}[\mathcal{X}]$. 
Example: the 1-Holed Torus

**Question.** What is the diagrammatic algebra corresponding to these trivalent $G$-trace diagrams??
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   - Generalizations
Trick to working with trace diagrams marked by representations:

1. Use injections $V_\lambda \hookrightarrow V \otimes \cdots \otimes V$ to access a “copy” of the representation lying inside tensor algebra (Young Projectors);
2. Use the $n$-Trace Diagram Calculus to manipulate the diagrams.

For many Lie groups, all irreducible representations can be understood in this way:

- $\text{SL}(2, \mathbb{C})$: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, etc.

- $\text{SL}(3, \mathbb{C})$: $\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$, etc.
Trick to working with trace diagrams marked by representations:

1. Use injections $V_\lambda \hookrightarrow V \otimes \cdots \otimes V$ to access a “copy” of the representation lying inside tensor algebra (*Young Projectors*);
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For many Lie groups, all irreducible representations can be understood in this way:

- $\text{SL}(2, \mathbb{C})$: \[ 2, \quad 3, \quad \text{etc.} \]
- $\text{SL}(3, \mathbb{C})$: \[ 3, \quad 3, \quad + \quad 3, \quad \text{etc.} \]
Trick to working with trace diagrams marked by representations:

1. Use injections $V_\lambda \hookrightarrow V \otimes \cdots \otimes V$ to access a “copy” of the representation lying inside tensor algebra (Young Projectors);
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For many Lie groups, all irreducible representations can be understood in this way:

- **SL(2, $\mathbb{C}$):** $\begin{array}{c} 2 \\ 3 \end{array}$, $\begin{array}{c} 3 \\ 3 \end{array}$, etc.
- **SL(3, $\mathbb{C}$):** $\begin{array}{c} 3 \\ 3 \end{array}$, $\begin{array}{c} 2 \\ 2 \end{array}$, $\begin{array}{c} 2 \\ 2 \end{array}$, etc.
Symmetrizers and Anti-Symmetrizers

**Definition**

The *symmetrizer* map \[ n \mapsto V^\otimes n \rightarrow V^\otimes n \] is the normalized sum of all permutations. For example:

\[
\begin{align*}
2 & = \frac{1}{2} (\mid \mid + X), \\
3 & = \frac{1}{6} (\mid \mid \mid + XX + XX + X \mid + XX + X \mid).
\end{align*}
\]

The *anti-symmetrizer* map \[ n \mapsto V^\otimes n \rightarrow V^\otimes n \] is the normalized sum of even permutations minus odd permutations. For example:

\[
\begin{align*}
2 & = \frac{1}{2} (\mid \mid - X), \\
3 & = \frac{1}{6} (\mid \mid \mid + XX + XX - X \mid - XX - X \mid).
\end{align*}
\]

The symmetrizer \[ n \mapsto V^\otimes n \] can be thought of as a map \( \text{Sym}_n(V) \hookrightarrow V^\otimes n \), hence picks out a copy of the irreducible representation \( V_n \) inside the tensor product.
Symmetrizers and Anti-Symmetrizers

Definition

The *symmetrizer* map $\begin{array}{ccc} n \end{array} : V \otimes n \to V \otimes n$ is the normalized sum of all permutations. For example:

\begin{align*}
\begin{array}{ccc} 2 \end{array} &= \frac{1}{2} \left( \begin{array}{c} | \ + X \end{array} \right), \\
\begin{array}{ccc} 3 \end{array} &= \frac{1}{6} \left( \begin{array}{c} | | | + XX + XX + X | + XX + | X \end{array} \right).
\end{align*}

The *anti-symmetrizer* map $\begin{array}{ccc} n \end{array} : V \otimes n \to V \otimes n$ is the normalized sum of even permutations minus odd permutations. For example:

\begin{align*}
\begin{array}{ccc} 2 \end{array} &= \frac{1}{2} \left( \begin{array}{c} | \ - X \end{array} \right), \\
\begin{array}{ccc} 3 \end{array} &= \frac{1}{6} \left( \begin{array}{c} | | | + XX + XX - X | - XX - | X \end{array} \right).
\end{align*}

The symmetrizer $\begin{array}{ccc} n \end{array}$ can be thought of as a map $\text{Sym}_n(V) \hookrightarrow V \otimes^n$, hence picks out a copy of the irreducible representation $V_n$ inside the tensor product.
When $G = SL(2, \mathbb{C})$, the following are true:

\[
\begin{align*}
\begin{array}{c}
\text{Diagram}
\end{array} & \quad \leftrightarrow \quad V_n = Sym_n(V); \\
\begin{array}{c}
\text{Diagram 2}
\end{array} & = \frac{1}{2} (\square - \chi) = \frac{1}{2} \bigtriangledown;
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Diagram n}
\end{array} & = 0 \quad \text{for } n > 2.
\end{align*}
\]

The symmetrizers can be rewritten as follows:

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 2}
\end{array} & = \frac{1}{2} (\square + \chi) = \square - \frac{1}{2} \bigtriangledown;
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 3}
\end{array} & = \square - \frac{1}{3} (\bigtriangleup + \bigtriangleup + \bigtriangleup) = \square - \frac{2}{3} (\bigtriangledown + \bigtriangledown) - \frac{1}{3} (\bigpentagon + \bigpentagon);
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 4}
\end{array} & = \square + \cdots.
\end{align*}
\]

This is the practical approach to the computation of central functions.
When $G = \text{SL}(2, \mathbb{C})$, the following are true:

$$
\begin{align*}
\begin{array}{c|c}
\hline
n & V_n = \text{Sym}_n(V) \\
\hline
2 & \frac{1}{2} (\big| - X) = \frac{1}{2} \bigcup \\
\hline
n & = 0 \quad \text{for } n > 2.
\end{array}
\end{align*}
$$

The symmetrizers can be rewritten as follows:

$$
\begin{align*}
\begin{array}{c|c}
\hline
2 & = \frac{1}{2} (\big| + X) = \big| - \frac{1}{2} \bigcup \\
\hline
3 & = \big| - \frac{1}{3} (\bigcup + \bigcup + \bigcup) = \big| - \frac{2}{3} (\bigcup + \bigcup) - \frac{1}{3} (\bigcup + \bigcup) \\
\hline
4 & = \big| + \cdots.
\end{array}
\end{align*}
$$

This is the practical approach to the computation of central functions.
Example

Compute $\chi_3(A) = \begin{array}{c}
\text{Diagram}
\end{array}^3$.

**Solution.** Expand the symmetrizer as follows:

$$\chi_3(A) = [A]^3 - \frac{1}{3} (3[A][A\bar{A}]) = [A]^3 - \frac{1}{3} (6[A]) = [A]^3 - 2[A].$$
Example

Compute \( \chi_3(A) = \begin{array} \end{array} \).

**Solution.** Expand the symmetrizer as follows:

\[
\begin{array} \end{array} = 3 - \frac{1}{3} \left( \begin{array} \end{array} + \begin{array} \end{array} + \begin{array} \end{array} \right).
\]

Apply the matrix and close off the terms to get:

\[
\chi_3(A) = [A]^3 - \frac{1}{3} (3[A][A\bar{A}]) = [A]^3 - \frac{1}{3} (6[A]) = [A]^3 - 2[A].
\]
Example

Compute $\chi_3(A) = \begin{array}{c}
  3
\end{array}^3$.

**Solution.** Expand the symmetrizer as follows:

$$
\begin{array}{c}
  3
\end{array} = \frac{1}{3} \left( \begin{array}{c}
  3
\end{array} + \begin{array}{c}
  3
\end{array} + \begin{array}{c}
  3
\end{array} \right).
$$

Apply the matrix and close off the terms to get:

$$
\chi_3(A) = [A]^3 - \frac{1}{3} (3[A][A\bar{A}]) = [A]^3 - \frac{1}{3} (6[A]) = [A]^3 - 2[A].
$$
Define $\Theta(a, b, c) = \begin{array}{c}
\sum
\end{array}$ and $\Delta(c) = \begin{array}{c}
\sum
\end{array}$. Then:

**Proposition (Bubble, Fusion Relations)**

\[
\begin{align*}
\begin{array}{c}
\sum
\end{array} &= \left( \frac{\Theta(a, b, c)}{\Delta(c)} \right) \delta_{cd}; \\
\begin{array}{c}
\sum
\end{array} &= \sum_{c \in [a,b]} \left( \frac{\Delta(c)}{\Theta(a, b, c)} \right) \begin{array}{c}
\sum
\end{array}.
\end{align*}
\]

**Proposition (Recoupling)**

\[
\begin{array}{c}
\sum
\end{array} = C \begin{array}{c}
\sum
\end{array}
\]

for some coefficient $C$ depending on $a, \ldots, f$ called a $6j$-Symbol.
Define $\Theta(a, b, c) = \begin{tikzpicture}[baseline=0.5ex]
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (2,0) {$c$};
\draw (a) edge (b) edge (c) (b) edge (c);
\end{tikzpicture}$ and $\Delta(c) = \begin{tikzpicture}[baseline=0.5ex]
\node (c) at (0,0) {$c$};
\end{tikzpicture}$. Then:

**Proposition (Bubble, Fusion Relations)**

\[ \begin{tikzpicture}[baseline=0.5ex]
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (d) at (2,0) {$d$};
\node (c) at (1,1) {$c$};
\draw (a) edge (b) edge (d) (b) edge (c) (d) edge (c);
\end{tikzpicture} = \left( \frac{\Theta(a, b, c)}{\Delta(c)} \right) \delta_{cd}; \]

\[ \begin{tikzpicture}[baseline=0.5ex]
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (0,1) {$c$};
\node (d) at (1,1) {$d$};
\draw (a) edge (b) edge (c) (b) edge (d) (c) edge (d);
\end{tikzpicture} = \sum_{c \in [a, b]} \left( \frac{\Delta(c)}{\Theta(a, b, c)} \right) \begin{tikzpicture}[baseline=0.5ex]
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (1,1) {$c$};
\draw (a) edge (b) edge (c) (b) edge (c);
\end{tikzpicture}; \]

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\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (2,0) {$c$};
\node (d) at (1,1) {$d$};
\node (e) at (0,1) {$e$};
\node (f) at (2,1) {$f$};
\draw (a) edge (b) edge (c) edge (d) (b) edge (e) edge (f) (c) edge (f);
\end{tikzpicture} = C \begin{tikzpicture}[baseline=0.5ex]
\node (a) at (0,0) {$a$};
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\node (f) at (2,1) {$f$};
\draw (a) edge (b) edge (c) edge (d) (b) edge (e) edge (f) (c) edge (f);
\end{tikzpicture} \]

for some coefficient $C$ depending on $a, \ldots, f$ called a $6j$-Symbol.
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Definition

The rank one central functions (corresponding to an annulus) are

$$\chi_a(A) = \textcircled{a}.$$
Proposition (Rank One Product Formula)

\[ \chi_a \cdot \chi_b = \sum_{c \in [a, b]} \chi_c \cdot \]

Proof.

The relation

\[ a \prod_{a} b = \sum_{c \in [a, b]} \left( \frac{\Delta(c)}{\Theta(a, b, c)} \right) a \, c, \, b \]

implies

\[ \prod_{a} a \prod_{b} b = \sum_{c \in [a, b]} \left( \frac{\Delta(c)}{\Theta(a, b, c)} \right) a \, \Theta(b, c) \, a \, b \, c. \]

Pull the matrix through the node and apply the bubble identity:

\[ \chi_a \chi_b = \sum_{c \in [a, b]} \left( \frac{\Delta(c)}{\Theta(a, b, c)} \right) \left( \frac{\Theta(a, b, c)}{\Delta(c)} \right) \]

\[ \prod_{a} c. \]
Proposition (Rank One Product Formula)

\[ \chi_a \cdot \chi_b = \sum_{c \in \{a, b\}} \chi_c. \]

Proof.

The relation

\[ \begin{array}{c}
\bigcirc^a \\
\bigcirc^b
\end{array} = \sum_{c \in \{a, b\}} \left( \frac{\Delta(c)}{\Theta(a, b, c)} \right) \begin{array}{c}
\bigcirc^a \\
\bigcirc^c
\end{array} \]

implies

\[ \begin{array}{c}
\bigcirc^a \\
\bigcirc^b
\end{array} = \sum_{c \in \{a, b\}} \left( \frac{\Delta(c)}{\Theta(a, b, c)} \right) \begin{array}{c}
\bigcirc^a \\
\bigcirc^c
\end{array}. \]

Pull the matrix through the node and apply the bubble identity:

\[ \chi_a \chi_b = \sum_{c \in \{a, b\}} \left( \frac{\Delta(c)}{\Theta(a, b, c)} \right) \left( \frac{\Theta(a, b, c)}{\Delta(c)} \right) \begin{array}{c}
\bigcirc^c
\end{array}. \]
Proposition (Rank One Product Formula)

\[ \chi_a \cdot \chi_b = \sum_{c \in [a, b]} \chi_c. \]

Proof.

The relation

\[ \begin{array}{c}
\begin{array}{c}
\sum_{c \in [a, b]} \\
\Delta(c)
\end{array} \cdot \\
\Theta(a, b, c)
\end{array} \]

implies

\[ \sum_{c \in [a, b]} \left( \frac{\Delta(c)}{\Theta(a, b, c)} \right) \]

Pull the matrix through the node and apply the bubble identity:

\[ \chi_a \chi_b = \sum_{c \in [a, b]} \left( \frac{\Delta(c)}{\Theta(a, b, c)} \right) \left( \frac{\Theta(a, b, c)}{\Delta(c)} \right) \]

\[ \square \]
As a special case of the previous result,

\[ x \cdot \chi_{a-1} = \chi_1 \cdot \chi_{a-1} = \chi_{a-2} + \chi_a. \]

Hence, the central functions satisfy the *Chebyshev* or *Fibonacci* recurrence \( \chi_a = x \cdot \chi_{a-1} - \chi_{a-2} \) and are easily computed:

\[
\begin{align*}
\chi_0(A) &= 1 \\
\chi_1(A) &= x \\
\chi_2(A) &= x^2 - 1 \\
\chi_3(A) &= x^3 - 2x \\
\chi_4(A) &= x^4 - 3x + 1.
\end{align*}
\]
As a special case of the previous result,

\[ x \cdot \chi_{a-1} = \chi_1 \cdot \chi_{a-1} = \chi_{a-2} + \chi_a. \]

Hence, the central functions satisfy the *Chebyshev* or *Fibonacci* recurrence \( \chi_a = x \cdot \chi_{a-1} - \chi_{a-2} \) and are easily computed:

\[
\begin{align*}
\chi_0(A) &= 1 \\
\chi_1(A) &= x \\
\chi_2(A) &= x^2 - 1 \\
\chi_3(A) &= x^3 - 2x \\
\chi_4(A) &= x^4 - 3x + 1.
\end{align*}
\]
Some other results for rank one central functions:

- \( x^n = \sum_{r=0}^{\lfloor n/2 \rfloor} \left( \binom{n}{r} - \binom{n}{r-1} \right) \chi_{n-2r} \), where \( \binom{n}{r} = 0 \) for \( r \leq 0 \);

- \( \chi_n = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \binom{n-r}{r} x^{n-2r} \);

- If \( \lambda \) is an eigenvalue of \( A \), then \( \chi_n = [n + 1]_\lambda \), the quantized integer with \( q = \lambda \);

- If \( \text{tr}(A) = i = \sqrt{-1} \), then \( \chi_n = i^n F_n \), where \( F_n \) is the \( nth \) Fibonacci number.
Some other results for rank one central functions:

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Some other results for rank one central functions:

- $x^n = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \binom{n}{r} - \binom{n}{r-1} \right) \chi_{n-2r}$, where $\binom{n}{r} = 0$ for $r \leq 0$;
- $\chi_n = \sum_{r=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^r \binom{n-r}{r} x^{n-2r}$;
- If $\lambda$ is an eigenvalue of $A$, then $\chi_n = [n + 1]_\lambda$, the quantized integer with $q = \lambda$;
- If $\text{tr}(A) = i = \sqrt{-1}$, then $\chi_n = i^n F_n$, where $F_n$ is the $n$th Fibonacci number.
Some other results for rank one central functions:

- \( x^n = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r} - \binom{n}{r-1} \chi_{n-2r} \), where \( \binom{n}{r} = 0 \) for \( r \leq 0 \);
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- If \( \text{tr}(A) = i = \sqrt{-1} \), then \( \chi_n = i^n F_n \), where \( F_n \) is the \( n \)th Fibonacci number.
Rank Two

Definition

The rank two central functions for $\mathbb{C}[X_2]$ are: $\chi_{c}^{a,b}(A, B) = \frac{1}{2}(a+b-c)$, where $\{a, b, c\}$ is any admissible triple.

Alternate Parameters: Define $\chi_{\alpha, \beta, \gamma} \equiv \chi_{c}^{a,b}$, where $\alpha \equiv \frac{1}{2}(-a+b+c)$, $\beta \equiv \frac{1}{2}(a-b+c)$, and $\gamma \equiv \frac{1}{2}(a+b-c)$. These are the number of each type of loop occurring when the symmetrizers are expanded. The admissibility condition is then $\alpha, \beta, \gamma \geq 0$.

We typically write $\chi_{c}^{a,b}$ in terms of $x = \text{tr}(A)$, $y = \text{tr}(B)$, $z = \text{tr}(A\bar{B})$. 
**Rank Two**

**Definition**

The rank two central functions for $\mathbb{C}[\mathcal{X}_2]$ are: $\chi^{a,b}_c(A, B) = \frac{1}{2} \left( \begin{array}{c} a \cr b \cr c \end{array} \right)$, where $\{a, b, c\}$ is any admissible triple.

**Alternate Parameters:** Define $\chi_{\alpha,\beta,\gamma} = \chi_{c}^{a,b}$, where $\alpha \equiv \frac{1}{2}(-a + b + c)$, $\beta \equiv \frac{1}{2}(a - b + c)$, and $\gamma \equiv \frac{1}{2}(a + b - c)$. These are the number of each type of loop occurring when the symmetrizers are expanded. The admissibility condition is then $\alpha, \beta, \gamma \geq 0$.

We typically write $\chi^{a,b}_c$ in terms of $x = \text{tr}(A)$, $y = \text{tr}(B)$, $z = \text{tr}(A\bar{B})$. 

---

Elisha Peterson

Diagrammatic Central Functions
Rank Two

Definition

The rank two central functions for $\mathbb{C}[X_2]$ are: $\chi_{c}^{a,b}(A, B) = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\overline{A} \\
\overline{B}
\end{array}
\end{array}
\end{array}$,

where $\{a, b, c\}$ is any admissible triple.

Alternate Parameters: Define $\chi_{\alpha,\beta,\gamma} = \chi_{c}^{a,b}$, where $\alpha \equiv \frac{1}{2}(-a + b + c)$, $\beta \equiv \frac{1}{2}(a - b + c)$, and $\gamma \equiv \frac{1}{2}(a + b - c)$. These are the number of each type of loop occurring when the symmetrizers are expanded. The admissibility condition is then $\alpha, \beta, \gamma \geq 0$.

We typically write $\chi_{c}^{a,b}$ in terms of $x = \text{tr}(A)$, $y = \text{tr}(B)$, $z = \text{tr}(\overline{A}\overline{B})$. 

Elisha Peterson  
Diagrammatic Central Functions
The rank two central functions for $\mathbb{C}[X_2]$ are: $\chi_{c}^{a,b}(A, B) = \begin{array}{c}

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We typically write $\chi_{c}^{a,b}$ in terms of $x = \text{tr}(A), y = \text{tr}(B), z = \text{tr}(A\bar{B})$. 
Example

Compute $\chi^{1,2}_3(A, B) = \begin{array}{c}
\begin{array}{c}
\gamma \\
2 \\
3 \\
\gamma \\
1 \\
B \\
A
\end{array}
\end{array}$.

Solution. Expand the symmetrizer as follows:

$$\begin{array}{c}
\begin{array}{c}
\gamma \\
3 \\
\gamma \\
1 \\
2 \\
B \\
A
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
\gamma \\
3 \\
\gamma \\
1 \\
2 \\
B \\
A
\end{array}
\end{array} - \frac{1}{3} \left( \begin{array}{c}
\begin{array}{c}
\gamma \\
3 \\
\gamma \\
1 \\
2 \\
B \\
A
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\gamma \\
3 \\
\gamma \\
1 \\
2 \\
B \\
A
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\gamma \\
3 \\
\gamma \\
1 \\
2 \\
B \\
A
\end{array}
\end{array} \right).$$

Apply the matrix and close off the terms to get:

$$\chi^{1,2}_3(A, B) = [A][B]^2 - \frac{1}{3}([AB][B] + [AB][B] + 2[A])$$
$$= [A][B]^2 - \frac{2}{3}([AB][B] + [A]).$$

Example

Compute \( \chi_{3}^{1,2}(A, B) = \). 

Solution. Expand the symmetrizer as follows:

\[
\begin{array}{c}
\chi_{3}^{1,2}(A, B) = [A][B]^2 - \frac{1}{3} ([A\bar{B}][B] + [A\bar{B}][B] + 2[A]) \\
= [A][B]^2 - \frac{2}{3} ([A\bar{B}][B] + [A]).
\end{array}
\]
Example

Compute $\chi^{1,2}_3(A, B) = A \otimes B^2 \otimes B^3$.

**Solution.** Expand the symmetrizer as follows:

$$\begin{array}{c}
3 & = & | | 1 - \frac{1}{3} \left( \langle 1 + \langle 2 + \langle 3 \right) \\
\end{array}$$

Apply the matrix and close off the terms to get:

$$\chi^{1,2}_3(A, B) = [A][B]^2 - \frac{1}{3} ([A\bar{B}][B] + [A\bar{B}][B] + 2[A])$$

$$= [A][B]^2 - \frac{2}{3} ([A\bar{B}][B] + [A]).$$
Rank Two: Symmetry Property

**Theorem**

*If $\sigma$ is any permutation on three letters, then*

$$\chi_{\sigma(\alpha,\beta,\gamma)}(\sigma(y, x, z)) = \chi_{\alpha,\beta,\gamma}(y, x, z).$$

**Proof.**
Theorem

If σ is any permutation on three letters, then

\[ \chi_{\sigma(\alpha,\beta,\gamma)}(\sigma(y, x, z)) = \chi_{\alpha,\beta,\gamma}(y, x, z). \]

Proof.
Define \( \hat{\chi}_{\alpha,\beta,\gamma} \equiv a!b!c!\chi_{\alpha,\beta,\gamma} \) and \( \delta \equiv \alpha + \beta + \gamma \) (called the \textit{rank}).
Define \( \bar{\alpha} \equiv \alpha + 1 \) and \( \underline{\alpha} \equiv \alpha - 1 \).

\textbf{Theorem (Rank Two Recursion)}

\[
\hat{\chi}_{\alpha,\beta,\gamma} = x \cdot ac \hat{\chi}_{\alpha,\underline{\beta},\gamma} - \gamma^2 \hat{\chi}_{\alpha,\beta,\underline{\gamma}} - \alpha^2 \hat{\chi}_{\alpha,\underline{\beta},\underline{\gamma}} - \delta^2 (\beta - 2)^2 \hat{\chi}_{\alpha,\underline{\beta},\underline{\gamma}}.
\]

\textbf{Proof Idea.} Use the fusion identity to join the terms \( x = \text{tr}(A) \) and \( \chi_{\alpha,\underline{\beta},\underline{\gamma}} \), and the bubble identity to reduce the result back to the standard form of central functions.

\textbf{Notes.}

- As a corollary, the leading term of \( \chi_{\alpha,\beta,\gamma} \) is \( x^\beta y^\alpha z^\gamma \).
- Together with the symmetry property, this provides an efficient technique for computing central functions.
Rank Two: Recursion Property

Define $\hat{\chi}_{\alpha,\beta,\gamma} \equiv a!b!c!\chi_{\alpha,\beta,\gamma}$ and $\delta \equiv \alpha + \beta + \gamma$ (called the rank). Define $\overline{\alpha} \equiv \alpha + 1$ and $\underline{\alpha} \equiv \alpha - 1$.

**Theorem (Rank Two Recursion)**

$$\hat{\chi}_{\alpha,\beta,\gamma} = x \cdot ac\hat{\chi}_{\alpha,\beta,\gamma} - \gamma^2 \hat{\chi}_{\overline{\alpha},\beta,\gamma} - \alpha^2 \hat{\chi}_{\alpha,\overline{\beta},\gamma} - \delta^2 (\beta - 2)^2 \hat{\chi}_{\alpha,\underline{\beta},\gamma}.$$ 

**Proof Idea.** Use the fusion identity to join the terms $x = \text{tr}(A)$ and $\chi_{\alpha,\beta,\gamma}$, and the bubble identity to reduce the result back to the standard form of central functions.

**Notes.**

- As a corollary, the leading term of $\chi_{\alpha,\beta,\gamma}$ is $x^\beta y^\alpha z^\gamma$.
- Together with the symmetry property, this provides an efficient technique for computing central functions.
Rank Two: Recursion Property

Define \( \hat{\chi}_{\alpha,\beta,\gamma} \equiv a!b!c!\chi_{\alpha,\beta,\gamma} \) and \( \delta \equiv \alpha + \beta + \gamma \) (called the rank).
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\[
\hat{\chi}_{\alpha,\beta,\gamma} = x \cdot ac \hat{\chi}_{\alpha,\beta,\gamma} - \gamma^2 \hat{\chi}_{\overline{\alpha},\beta,\gamma} - \alpha^2 \hat{\chi}_{\alpha,\underline{\beta},\gamma} - \delta^2 (\beta - 2)^2 \hat{\chi}_{\alpha,\beta,\gamma}.
\]

Proof Idea. Use the fusion identity to join the terms \( x = tr(A) \) and \( \chi_{\alpha,\beta,\gamma} \), and the bubble identity to reduce the result back to the standard form of central functions.

Notes.

- As a corollary, the leading term of \( \chi_{\alpha,\beta,\gamma} \) is \( x^\beta y^{\alpha} z^{\gamma} \).
- Together with the symmetry property, this provides an efficient technique for computing central functions.
Rank Two: Table

\[ \delta = 0 \]

\[ \delta = 1 \]

\[ \delta = 2 \]

\[ \delta = 3 \]

\[ x^3 - 2x \]

\[ x^2 y - \frac{2}{3}(y + xz) \]

\[ x^2 y - \frac{2}{3}(y + xz) \]

\[ y^2 z + \ldots \]

\[ y^2 z + \ldots \]

\[ y^2 z + \ldots \]

\[ y^2 z + \ldots \]

\[ y^3 - 2y \]

\[ xy^2 - \frac{2}{3}(x + yz) \]

\[ x^2 - 1 \]

\[ x^2 - 1 \]

\[ yz - 2x \]

\[ xz - 2y \]

\[ yz^2 + \ldots \]

\[ xz^2 + \ldots \]

\[ \delta = 0 \]

\[ \delta = 1 \]

\[ \delta = 2 \]

\[ \delta = 3 \]
Theorem (Rank Two Product)

\[
\chi_{c}^{a,b} \chi_{c'}^{a',b'} = \sum_{j_1,j_2,k,l,m} C_{j_1,k,l,m} C_{j_2,k,l,m} \frac{\Theta(a,a',k)\Theta(b,b',l)\Theta(c,c',m)}{\Delta(k)\Delta(l)\Delta(m)} \chi_{k}^{l,m},
\]

where the sum is over eight admissible triples \(\{a, a', k\}, \ldots, \{k, l, m\}\) and

\[
C_{j_1,k,l,m} = \frac{\Delta(j_i)}{\Theta(a',b,j_i)} \left[ a \quad a' \quad k \right] \left[ b \quad b' \quad l \right] \left[ k \quad l \quad m \right] .
\]

Here, \([\cdots]\) are recoupling coefficients known as 6j Symbols.

Idea of Proof.
Theorem (Rank Two Product)

\[ \chi_c^{a,b} \chi_{c'}^{a',b'} = \sum_{j_1,j_2,k,l,m} C_{j_1,k,l,m} C_{j_2,k,l,m} \frac{\Theta(a,a',k)\Theta(b,b',l)\Theta(c,c',m)}{\Delta(k)\Delta(l)\Delta(m)} \chi_k^{l,m}, \]

where the sum is over eight admissible triples \( \{a, a', k\}, \ldots, \{k, l, m\} \) and

\[ C_{j_i,k,l,m} = \frac{\Delta(j_i)}{\Theta(a',b,j_i)} \left[ a \ a' \ k \right] \left[ b \ b' \ l \right] \left[ k \ l \ m \right]. \]

Here, \( \cdots \) are recoupling coefficients known as \( 6j \) Symbols.

Idea of Proof.
Theorem (Rank Two Product)

\[
\chi_{c,a,b} \chi_{c',a',b'} = \sum_{j_1,j_2,k,l,m} C_{j_1,k,l,m} C_{j_2,k,l,m} \frac{\Theta(a,a',k) \Theta(b,b',l) \Theta(c,c',m)}{\Delta(k) \Delta(l) \Delta(m)} \chi_{k,l,m},
\]

where the sum is over eight admissible triples \{a, a', k\}, \ldots, \{k, l, m\} and

\[
C_{j_i,k,l,m} = \frac{\Delta(j_i)}{\Theta(a',b,j_i)} \begin{bmatrix} a & a' & k \\ j_i & c & b \end{bmatrix} \begin{bmatrix} b & b' & l \\ j_i & c' & a' \end{bmatrix} \begin{bmatrix} k & l & m \\ c & c' & j_i \end{bmatrix}.
\]

Here, \[ \cdots \] are recoupling coefficients known as 6j Symbols.

Idea of Proof.
**Theorem (Rank Two Product)**

\[
\chi_{c}^{a,b} \chi_{c'}^{a',b'} = \sum_{j_1, j_2, k, l, m} C_{j_1, k, l, m} C_{j_2, k, l, m} \frac{\Theta(a, a', k) \Theta(b, b', l) \Theta(c, c', m)}{\Delta(k) \Delta(l) \Delta(m)} \chi_{k}^{l, m},
\]

where the sum is over eight admissible triples \{a, a', k\}, \ldots, \{k, l, m\} and

\[
C_{j_i, k, l, m} = \frac{\Delta(j_i)}{\Theta(a', b, j_i)} \left[ \begin{array}{ccc} a & a' & k \\ c & b & j_i \end{array} \right] \left[ \begin{array}{ccc} b & b' & l \\ c' & a' & j_i \end{array} \right] \left[ \begin{array}{ccc} k & l & m \\ c' & c & j_i \end{array} \right].
\]

Here, \([\cdots]\) are recoupling coefficients known as 6j Symbols.

**Idea of Proof.**

[Diagram showing the proof process with recoupling coefficients represented by 6j Symbols]
Theorem (Rank Two Product)

\[
\chi^{a,b}_{c} \chi^{a',b'}_{c'} = \sum_{j_1, j_2, k, l, m} C_{j_1, k, l, m} C_{j_2, k, l, m} \frac{\Theta(a, a', k) \Theta(b, b', l) \Theta(c, c', m)}{\Delta(k) \Delta(l) \Delta(m)} \chi^l_{m},
\]

where the sum is over eight admissible triples \{a, a', k\}, \ldots, \{k, l, m\} and

\[
C_{j_i, k, l, m} = \frac{\Delta(j_i)}{\Theta(a', b, j_i)} \left[\begin{array}{c} a & a' & k \\ j_i & c & b \end{array}\right] \left[\begin{array}{c} b & b' & l \\ j_i & c' & a' \end{array}\right] \left[\begin{array}{c} k & l & m \\ c' & c & j_i \end{array}\right].
\]

Here, \[
\left[\cdots\right]
\]
are recoupling coefficients known as 6j Symbols.

Idea of Proof.
Theorem (Rank Two Product)

\[ \chi_{c}^{a,b} \chi_{c'}^{a',b'} = \sum_{j_1j_2,k,l,m} C_{j_1,k,l,m} C_{j_2,k,l,m} \frac{\Theta(a,a',k)\Theta(b,b',l)\Theta(c,c',m)}{\Delta(k)\Delta(l)\Delta(m)} \chi_{k+l+m} \]

where the sum is over eight admissible triples \{a, a', k\}, \ldots, \{k, l, m\} and

\[ C_{j_i,k,l,m} = \frac{\Delta(j_i)}{\Theta(a',b,j_i)} \begin{bmatrix} a & a' & k \\ j_i & c & b \end{bmatrix} \begin{bmatrix} b & b' & l \\ j_i & c' & a' \end{bmatrix} \begin{bmatrix} k & l & m \\ c' & c & j_i \end{bmatrix}. \]

Here, \[\cdots\] are recoupling coefficients known as 6j Symbols.

Idea of Proof.

[Diagram showing the transformation of the central functions using 6j Symbols]
Definition

The rank three central functions for \( \mathbb{C}[\mathcal{X}_3] \) are:

\[
\chi_{a,b,c}^{d,e,f}(A, B, C) = \begin{array}{c}
\text{(Diagram)}
\end{array}
\]

where the triples \( \{a, b, d\}, \{a, b, e\}, \{c, d, f\}, \) and \( \{c, e, f\} \) are all admissible.

Remark. There are many choices of diagram for rank three central functions, and the polynomials obtained will be very different depending on how they are drawn.
Definition

The rank three central functions for $\mathbb{C}[x_3]$ are:

$$\chi^{a,b,c}_{d,e,f}(A, B, C) = \ldots,$$

where the triples $\{a, b, d\}$, $\{a, b, e\}$, $\{c, d, f\}$, and $\{c, e, f\}$ are all admissible.

Remark. There are many choices of diagram for rank three central functions, and the polynomials obtained will be very different depending on how they are drawn.
Example

Compute $\chi_{0,2,1}^{1,1,1} = \begin{array}{ccc} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & A & C \end{array}$. 
The interesting cases are when $a, b, c \neq 0$; otherwise, the functions reduce to rank two central functions. Also, either $d \neq 0$ or $e \neq 0$; otherwise, the diagram is disconnected.

Let $x = \text{tr}(A)$, $y = \text{tr}(B)$, $z = \text{tr}(C)$, $X = \text{tr}(B\tilde{C})$, $Y = \text{tr}(A\tilde{C})$, and $Z = \text{tr}(A\tilde{B})$.

**Case $a = b = c = 1$:**

- $\chi_{0,2,1}^{1,1,1} = \frac{1}{2}zZ - X$
- $\chi_{2,0,1}^{1,1,1} = \frac{1}{2}zZ - [AC\tilde{B}]$
- $\chi_{2,2,1}^{1,1,1} = xX - \frac{1}{2}(X + [AC\tilde{B}]) + \frac{1}{4}zZ$
- $\chi_{2,2,3}^{1,1,1} = xyz - \frac{2}{3}(zZ + xX) + \frac{1}{3}(X + [AC\tilde{B}])$.

**Case $a = b = c = 2$:**

- $\chi_{2,2,2}^{2,2,2} = xXzZ - \frac{1}{2}(xyZ + xYz + Xyz + XYZ + XzZ) + \frac{1}{4}(x^2 + X^2 + y^2 + Y^2) + \frac{1}{2}(z^2 + Z^2) + \frac{1}{4}z^2Z^2 - \frac{1}{2}zZ[AC\tilde{B}] - 1$
The interesting cases are when $a, b, c \neq 0$; otherwise, the functions reduce to rank two central functions. Also, either $d \neq 0$ or $e \neq 0$; otherwise, the diagram is disconnected.

Let $x = \text{tr}(A)$, $y = \text{tr}(B)$, $z = \text{tr}(C)$, $X = \text{tr}(B\bar{C})$, $Y = \text{tr}(A\bar{C})$, and $Z = \text{tr}(A\bar{B})$.

**Case** $a = b = c = 1$:

- $\chi_{0,2,1}^{1,1,1} = \frac{1}{2}zZ - X$
- $\chi_{2,0,1}^{1,1,1} = \frac{1}{2}zZ - [AC\bar{B}]$
- $\chi_{2,2,1}^{1,1,1} = xX - \frac{1}{2}(X + [AC\bar{B}]) + \frac{1}{4}zZ$
- $\chi_{2,2,3}^{1,1,1} = xyz - \frac{2}{3}(zZ + xX) + \frac{1}{3}(X + [AC\bar{B}])$.

**Case** $a = b = c = 2$:

- $\chi_{2,2,2}^{2,2,2} = xXzZ - \frac{1}{2}(xyZ + xYZ + Xyz + XYZ + XzZ) + \frac{1}{4}(x^2 + X^2 + y^2 + Y^2) + \frac{1}{2}(z^2 + Z^2) + \frac{1}{4}z^2Z^2 - \frac{1}{2}zZ[AC\bar{B}] - 1$
The interesting cases are when \( a, b, c \neq 0 \); otherwise, the functions reduce to rank two central functions. Also, either \( d \neq 0 \) or \( e \neq 0 \); otherwise, the diagram is disconnected.

Let \( x = \text{tr}(A) \), \( y = \text{tr}(B) \), \( z = \text{tr}(C) \), \( X = \text{tr}(B\bar{C}) \), \( Y = \text{tr}(A\bar{C}) \), and \( Z = \text{tr}(A\bar{B}) \).

**Case \( a = b = c = 1 \):**

- \( \chi_{0,2,1}^{1,1,1} = \frac{1}{2} zZ - X \)
- \( \chi_{2,0,1}^{1,1,1} = \frac{1}{2} zZ - [AC\bar{B}] \)
- \( \chi_{2,2,1}^{1,1,1} = xX - \frac{1}{2}(X + [AC\bar{B}]) + \frac{1}{4}zZ \)
- \( \chi_{2,2,3}^{1,1,1} = xyz - \frac{2}{3}(zZ + xX) + \frac{1}{3}(X + [AC\bar{B}]). \)

**Case \( a = b = c = 2 \):**

- \( \chi_{2,2,2}^{2,2,2} = xXzZ - \frac{1}{2}(xyZ + xYz + Xyz + XYZ + XzZ) + \frac{1}{4}(x^2 + X^2 + y^2 + Y^2) + \frac{1}{2}(z^2 + Z^2) + \frac{1}{4}z^2Z^2 - \frac{1}{2}zZ[AC\bar{B}] - 1 \)
1. The Central Function Basis
   - Algebraic Approach
   - Diagrammatic Approach

2. Trace Diagrams and Representation Theory
   - Representations and Tensor Algebra
   - $SL(2, \mathbb{C})$ Trivalent Diagrams

3. Computation of Central Functions
   - Rank One
   - Rank Two
   - Rank Three

4. Questions for Exploration
   - Computing $SL(2, \mathbb{C})$ Central Functions
   - Generalizations
Central Functions for Rank Three and Beyond

There are a number of challenges to overcome before the general theory of central functions can be fully developed. For a systematic approach to work, central functions should be defined in a standard, symmetric way for all ranks. There are many ways to do this, for example:

\[ \begin{array}{c}
A \\
B \\
C
\end{array} \quad \text{and} \quad \begin{array}{c}
A \\
B \\
C
\end{array} \]

The advantage in either case is that the functions are built up from several identical components. This should provide a standard technique for developing recursion and product formulas, as well as computational algorithms.
Central Functions and Surface Structure

**Question.** How can trace diagrams take into account the structure of the *surface* as well as its fundamental group?

**Partial Answer.** Use the Poisson structure!

**Definition**

The Goldman bracket \( \{ f, g \} \) of two loops on a surface is the sum over all essential intersections of the following:

\[
\begin{array}{c}
\int & \rightarrow & \int \leftarrow \\
\int \times & \rightarrow & \int \times
\end{array}
\]

- This bracket satisfies the Jacobi and Leibniz identities, and so gives the ring a Poisson structure.
- The bracket is simply the application of the binor identity \( \int = \int \leftarrow \int \rightarrow \int \) to the essential crossings in a diagram.
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Definition

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\[
\begin{align*}
\begin{array}{c}
\times \\
\times
\end{array}
\quad \rightarrow \\
\begin{array}{c}
\rightarrow \\

\end{array}
\end{align*}
\]

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- The bracket is simply the application of the binor identity \( \begin{array}{c}
\times \\
\times
\end{array} = \begin{array}{c}
\rightarrow \\

\end{array} \) to the essential crossings in a diagram.
To generalize to $n \times n$ matrices, generalize $\bigcup$:

**Example**

Trace diagrams for $3 \times 3$ matrices are trivalent graphs, with antisymmetrizer

$$
\bigcup = 6 \frac{3}{3} = \big| \big| + \big| \big| \big| + \big| \big| - \big| \big| - \big| \big| - \big| \big| - \big| \big| - \big| \big| - \big| \big|.
$$

The local maxima and minima are defined via

$$
\bigcup : \mathbb{C} \to V \otimes V \otimes V;
$$

$$
1 \mapsto e_1 \otimes e_2 \otimes e_3 + \cdots - e_1 \otimes e_3 \otimes e_2.
$$

Predrag Cvitanovic (a physicist) has generalized the SL$(2, \mathbb{C})$ diagrams, known to physicists as “spin networks” to all classical Lie groups.
Generalizations: Higher Dimensions

To generalize to $n \times n$ matrices, generalize $\bigcup$:

**Example**

Trace diagrams for $3 \times 3$ matrices are trivalent graphs, with antisymmetrizer

$$\bigcup = 6 \begin{tikzpicture}[baseline=-0.5em]
\draw (0,0) -- (0.5,0) -- (0.5,0.5) -- (0,0);
\end{tikzpicture} = \begin{array}{c}
\v\v\v
\end{array} + \begin{array}{c}
\times\times
\end{array} + \begin{array}{c}
\times\times
\end{array} - \begin{array}{c}
\times
\end{array} - \begin{array}{c}
\times
\end{array} - \begin{array}{c}
\times
\end{array}.$$

The local maxima and minima are defined via

$$\bigcup : \mathbb{C} \rightarrow V \otimes V \otimes V;$$

$$1 \mapsto e_1 \otimes e_2 \otimes e_3 + \cdots - e_1 \otimes e_3 \otimes e_2.$$

Predrag Cvitanovic (a physicist) has generalized the $SL(2, \mathbb{C})$ diagrams, known to physicists as “spin networks” to all classical Lie groups.
**Question.** Can the theory of central functions be developed using Lie algebras rather than Lie groups?

**Answer.** Yes... I think! One interesting fact is that there is a nice diagram for transforming a matrix \( X \in \text{SL}(2, \mathbb{C}) \) into a matrix \( x \in \mathfrak{sl}(2, \mathbb{C}) \), since the result necessarily has trace 0. This is the mapping

\[
X \mapsto X \frac{1}{2} \text{tr}(X)/.
\]

Such diagrams are the primary “building blocks” of one type of central function:
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Central Functions and Quantum Groups

**Question.** How does this all relate to knot theory and quantum groups?

**Answer.**

- Central functions are very closely related to the theory obtained from “quantizing” crossings. The correspondence is exact in rank one.
- Is there a quantum version of central functions?
- The quantization of the trace diagram algebra is the *Kauffman Bracket Skein Module* of a surface.
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Acknowledgments/References

- Sean Lawton
- Bill Goldman
- Charlie Frohman
- Predrag Cvitanovic, *Group Theory*,
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- Carter/Flath/Saito, *The Classical and Quantum 6j-Symbols*