Exceptional Dehn fillings

July 9, 2007

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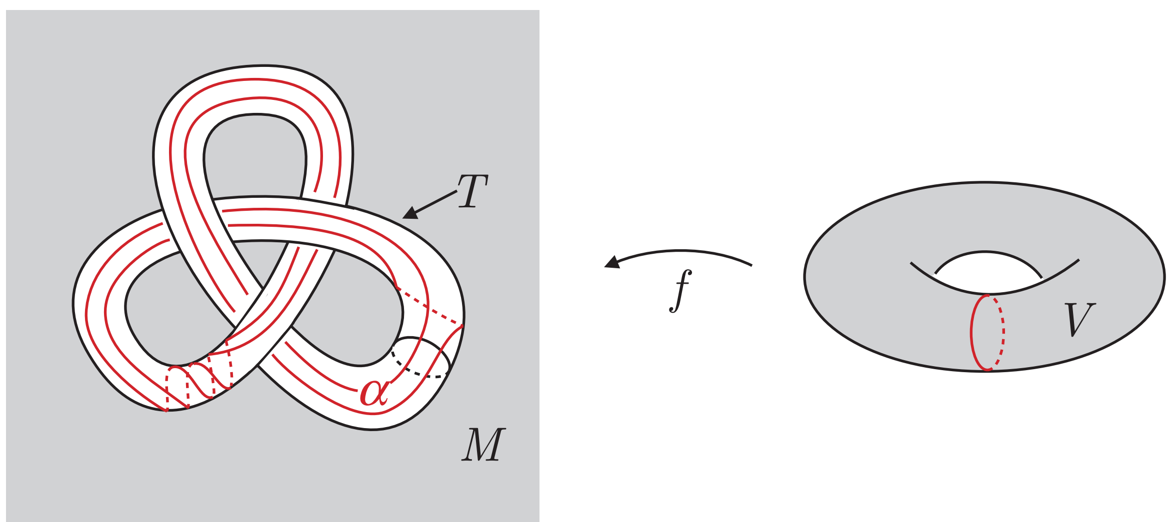
Dehn fillings

M is a compact, connected, orientable 3-manifold with a torus boundary component T.



$$M(\alpha) = M \cup_f V$$

A *slope* is the isotopy class of an essential circle on T ($T \subset \partial M$).

A compact orientable 3-manifold M is *hyperbolic* if M with its boundary tori removed has a finite volume complete hyperbolic structure.

Theorem (Hyperbolic Dehn Surgery Theorem). *If M is a hyperbolic 3-manifold with a torus boundary component T, then $M(\alpha)$ are hyperbolic for all but finitely many slopes α on T.*

Research Aim

M : hyperbolic $\longrightarrow M(\alpha)$: not hyperbolic for finitely many slopes

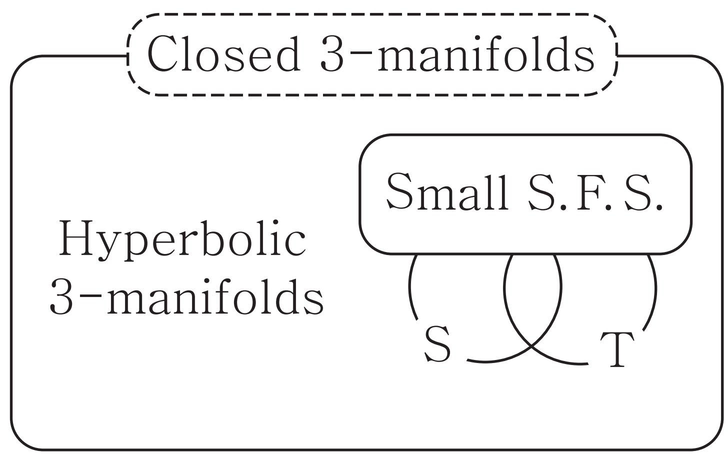
Such slopes are called *exceptional slopes*.

Project How many exceptional slopes?

Example The figure-8 knot exterior has 10 exceptional slopes.

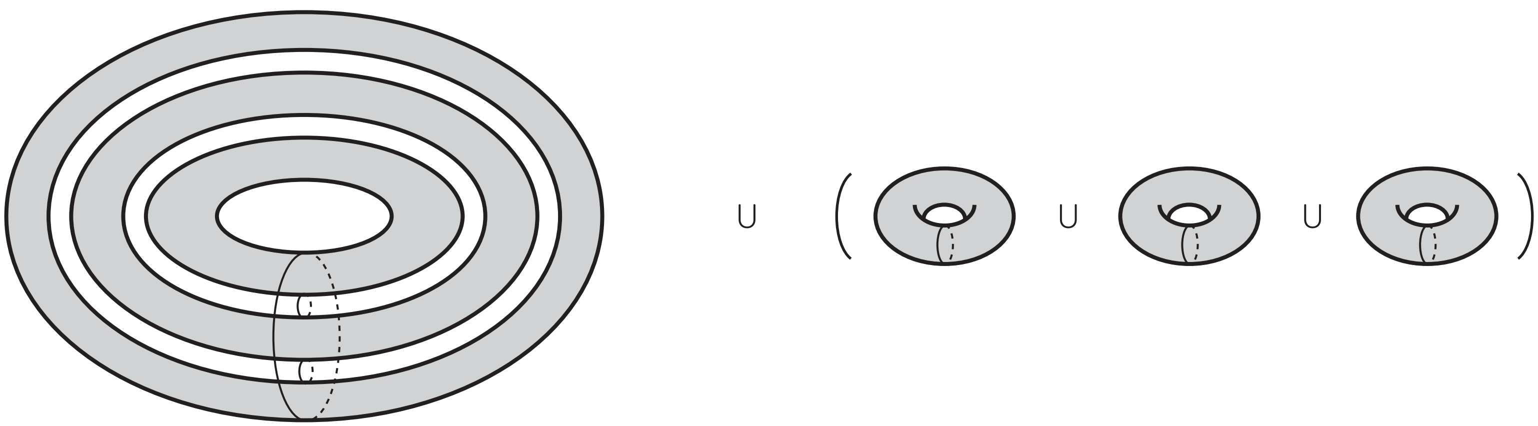
Conjecture (Gordon). There are at most 8 exceptional slopes for hyperbolic 3-manifolds except the figure-8 knot exterior.

Geometrization Conjecture A closed 3-manifold is not hyperbolic if and only if it is reducible, toroidal, or a small Seifert fibered space.



Small Seifert fibered spaces

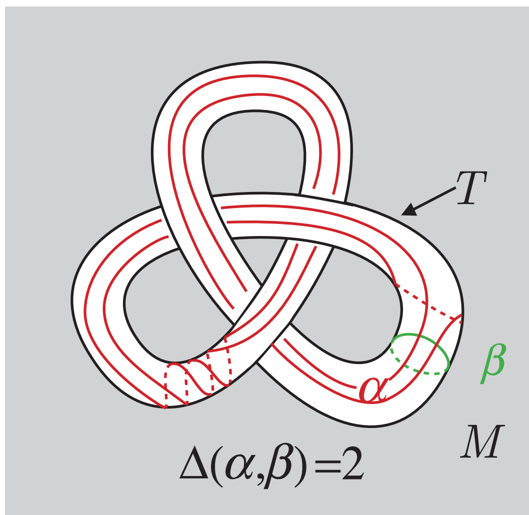
Small Seifert fibered spaces are obtained from $P \times S^1$ by suitably performing Dehn filling 3 times, where P is a pair of pants.



Distance between two slopes

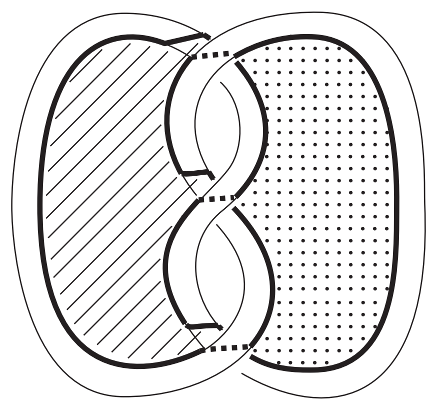
Let α, β be two slopes on $T \subset \partial M$.
$\Delta(\alpha, \beta) :=$ minimal geometric intersection number of α and β.

Example

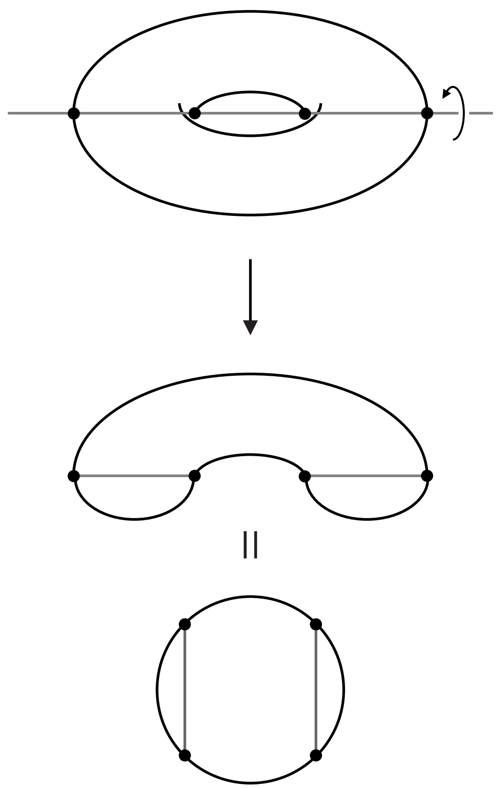


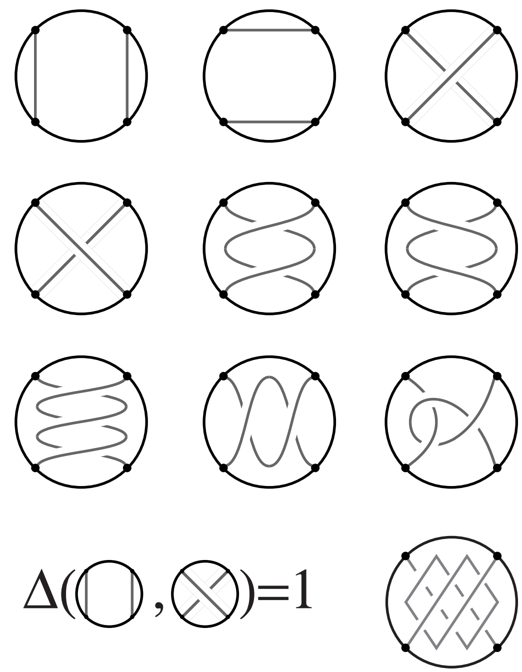
Parameterizing slopes

K : a knot in S^3, $M_K = S^3 - \text{int}N(K)$
μ, λ : meridian and longitude $\subset \partial M_K$.
α : an essential simple closed curve on ∂M_K
$\alpha \sim m\mu + l\lambda$ for some coprime integers m, l.
$\{\text{Slopes}\} \leftrightarrow \mathbb{Q} \cup \{1/0\}$, $\alpha \leftrightarrow m/l$
$\Delta(p/q\text{-slope}, r/s\text{-slope}) = |ps - qr|$

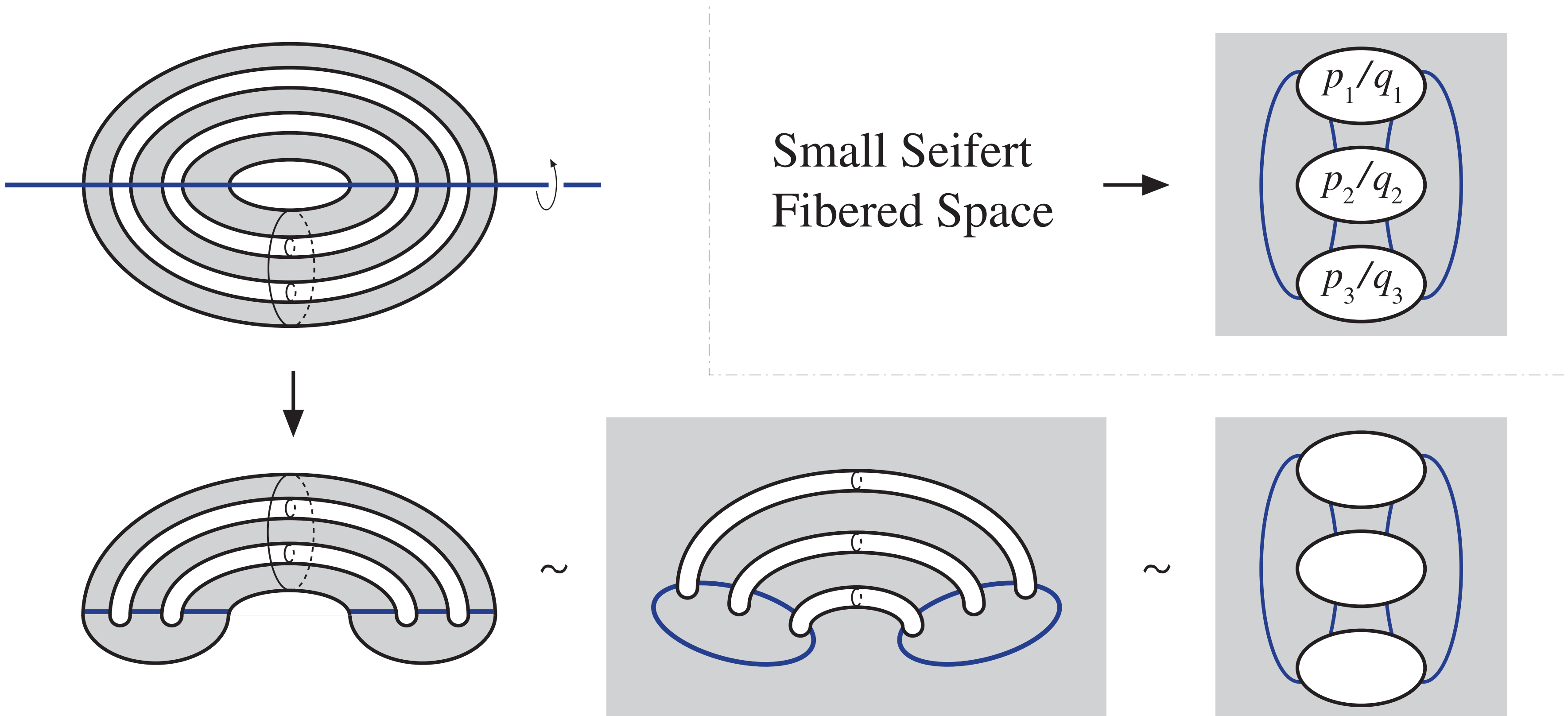


Double branched coverings and rational tangles





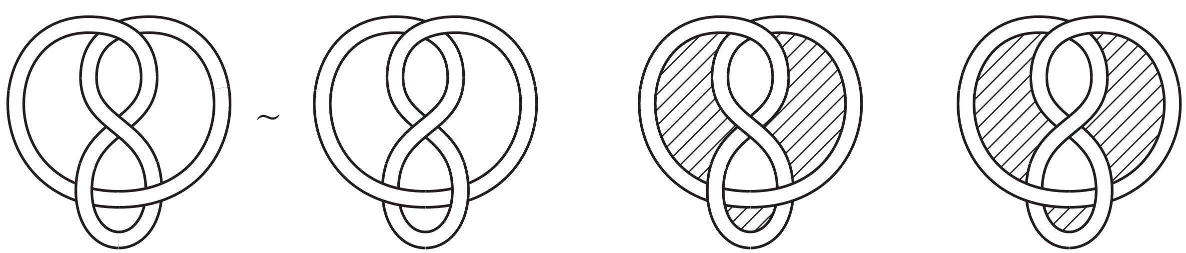
$$\Delta(p/q, r/s) = |ps - qr|$$



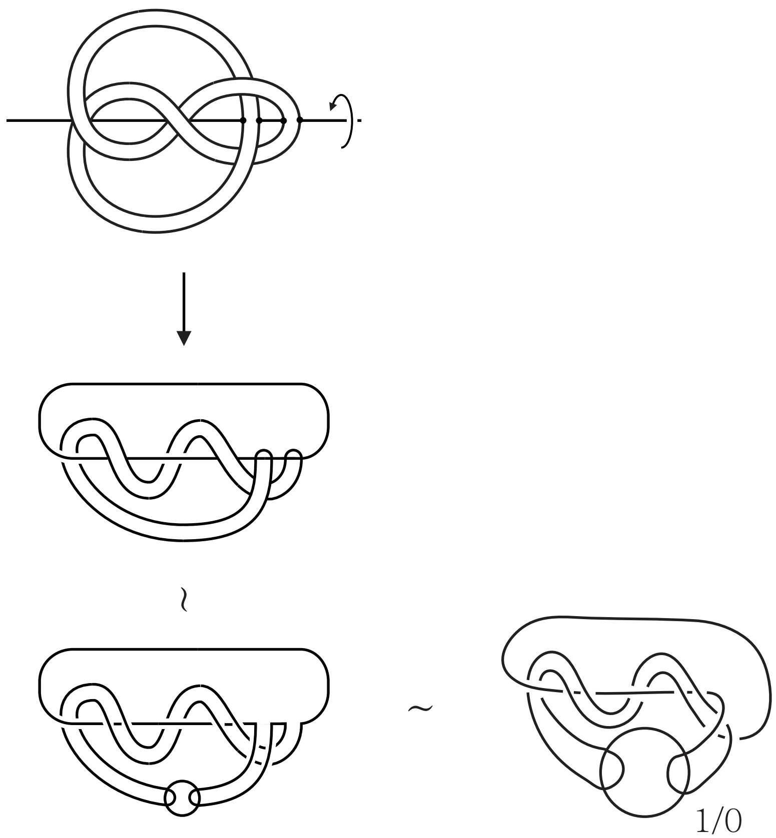
Small Seifert
Fibered Space
p_1/q_1
p_2/q_2
p_3/q_3
~
~

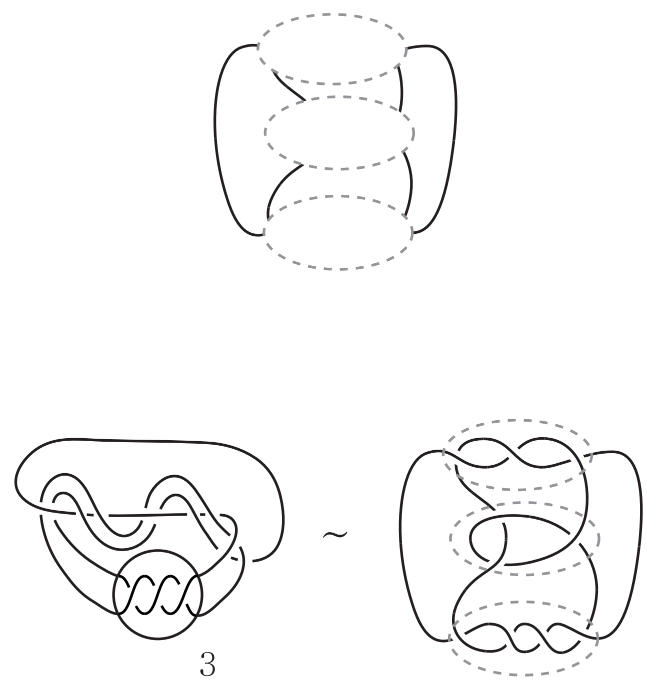
The figure-8 knot exterior and exceptional slopes

Let M be the exterior of the figure-8 knot.
Then $\mathcal{E}(M) = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 1/0\}$
Since the figure-8 knot is amphicheiral, $M(r) \cong M(-r)$.



boundary slope 4 boundary slope -4 $\Delta(4,-4)=8$


1/0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A closed 3-manifold is *hyperbolike* if it is irreducible, atoroidal, and is not a small Seifert fibered space.

Let M be a hyperbolic 3-manifold with a torus boundary component T. Define

$$\mathcal{E}(M;T) = \mathcal{E}(M) = \{\alpha \subset T \mid M(\alpha) \text{ is not hyperbolic } \}$$

$$\mathcal{E}'(M;T) = \mathcal{E}'(M) = \{\alpha \subset T \mid M(\alpha) \text{ is not hyperbolike } \} \subset \mathcal{E}(M).$$

Then Gordon's conjecture is reformulated as follows.

Conjecture $|\mathcal{E}(M)| \leq 8$ if M is not the figure-8 knot exterior.

Known results

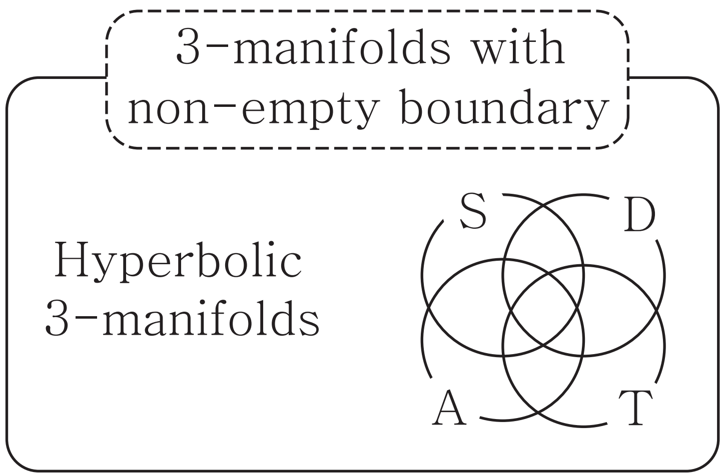
Theorem (Bleiler-Hodgson). *Let M be a hyperbolic 3-manifold with ∂M a single torus. Then $|\mathcal{E}'(M)| \leq 24$.*

Agol and Lackenby greatly improved this estimation.

Theorem (Agol, Lackenby). $|\mathcal{E}'(M)| \leq 12$.

What if ∂M is not a single torus?

Theorem (Geometrization Theorem for Haken manifolds). *A compact 3-manifold with non-empty boundary is not hyperbolic if and only if it is reducible($\mathcal{S}$), boundary-reducible ($\mathcal{D}$), annular ($\mathcal{A}$), or toroidal ($\mathcal{T}$).*



Known results (continued)

$\Delta \leq ?$	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	1	0	2	3
$\mathcal{D}$		1	2	2
$\mathcal{A}$			5	5
$\mathcal{T}$				8

Upper bounds for Δ

For example, $\Delta(\mathcal{S}, \mathcal{A}) \leq 2$ means:
Given a hyperbolic manifold M, if $M(\alpha), M(\beta)$ each contain an essential sphere and an essential annulus, then $\Delta(\alpha, \beta) \leq 2$ [Wu, Qiu].

$\Delta \leq ?$	0	1	2	3	4	5	6	7	8
$\sharp\{\text{slopes}\} \leq ?$	1	3	4	6	6	8	8	10	12

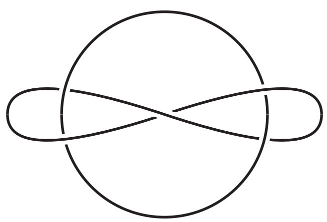
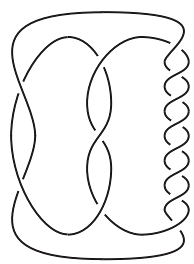
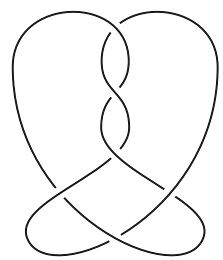
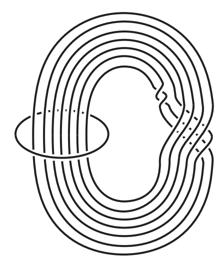
Theorem (Gordon). *Let M be a hyperbolic 3-manifold such that both $M(\alpha)$ and $M(\beta)$ contain an essential torus. Then either*
(1) $\Delta(\alpha, \beta) \leq 5$*; or*
(2) $\Delta(\alpha, \beta) = 6$ *and* $M \cong W(-2)$*; or*
(3) $\Delta(\alpha, \beta) = 7$ *and* $M \cong W(5/2)$*; or*
(4) $\Delta(\alpha, \beta) = 8$ *and* $M \cong W(-1)$ *or* $M \cong W(5)$*.*

Therefore in general, $|\mathcal{E}(M)| \leq 8$ for hyperbolic 3-manifolds M with at least two boundary components.

Examples

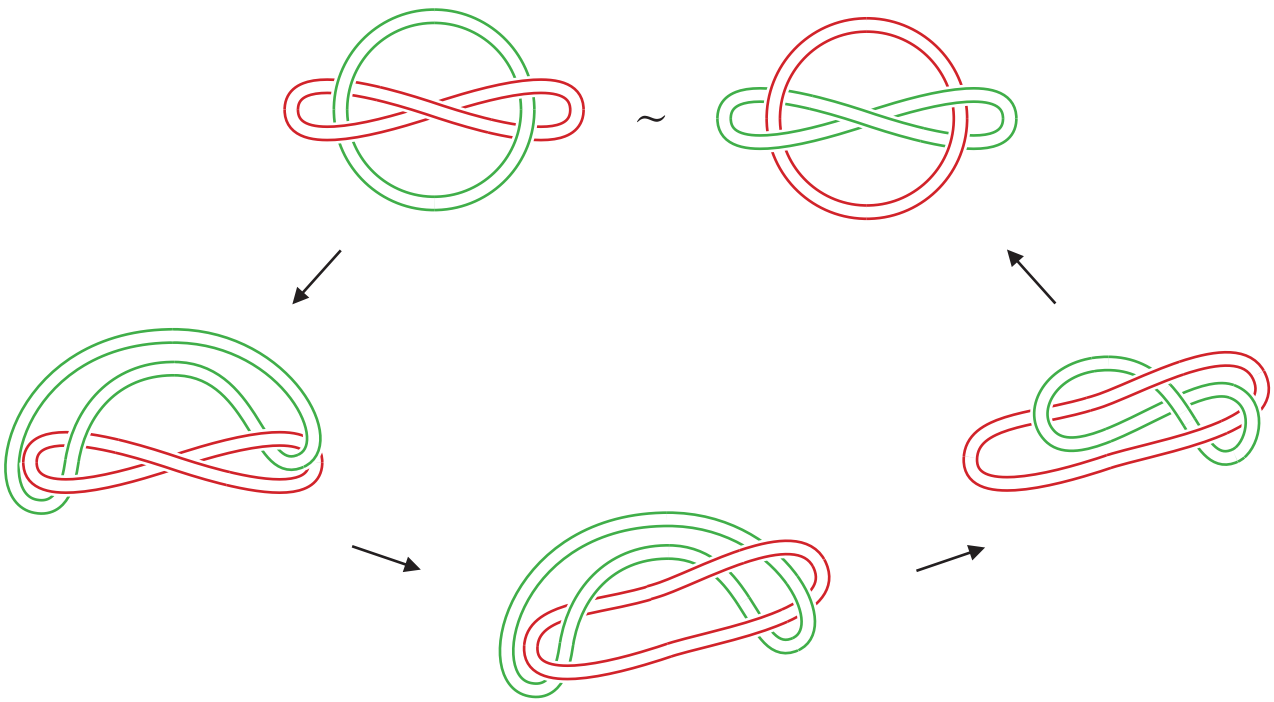
For hyperbolic 3-manifolds M with at least two boundary components, the maximal observed value for $|\mathcal{E}(M)|$ is 6.

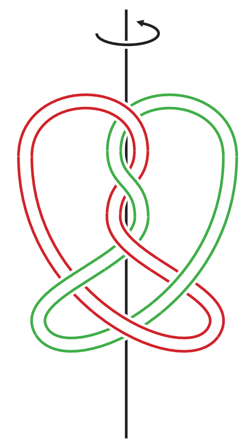
The following links are the Whitehead link, the Whitehead sister link, the 2-bridge link associated to 3/10 in Conway's notation, and the Berge link.

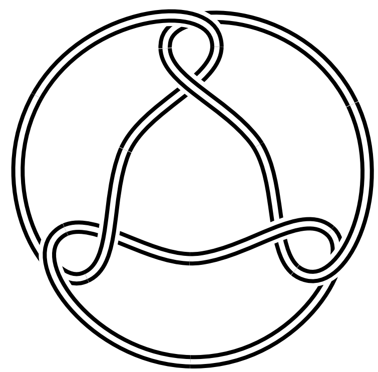
Theorem (Martelli-Petronio). *Their exteriors have exactly* $\mathbf{6}$ *exceptional slopes.*

Let M_1, M_2, M_3, M_4 be the exteriors of the above links, respectively. Then there is a self-homeomorphism of M_i interchanging two boundary tori, $i = 1, 2, 3, 4$. For M_1 and M_3, see the following figure.





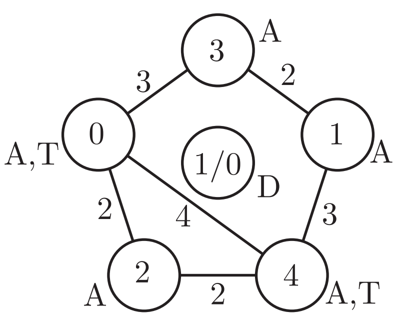
In fact, these manifolds are obtained from the exterior of the below link by Dehn filling.



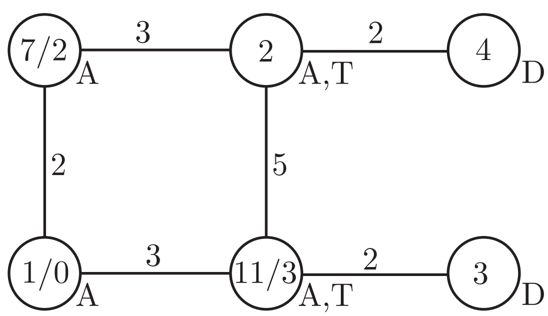
Hence, there is the desired homeomorphism for each M_i.

Exceptional slopes for the link exteriors

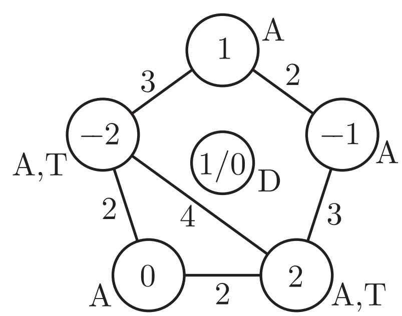
The exceptional slopes for M_1, M_2, M_3, M_4 and the mutual distances are shown below. (Unlabelled distances are 1.)



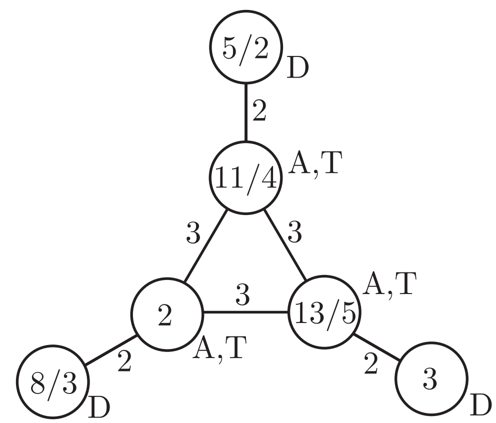
exceptional slopes for M_1



exceptional slopes for M_2



exceptional slopes for M_3



exceptional slopes for M_4

Theorem (Frigerio-Martelli-Petronio). *For any* $g \geq 2$*, there are infinitely many hyperbolic* 3*-manifolds with* $\mathbf{6}$ *exceptional Dehn fillings among which three yield a handlebody of genus* g *and the others yield an annular manifold.*

We remark that these manifolds are built by generalizing the Berge link exterior.

Theorem (Lee). *Let* M *be a hyperbolic* 3*-manifold with one torus boundary component and at least one other boundary component. Then*

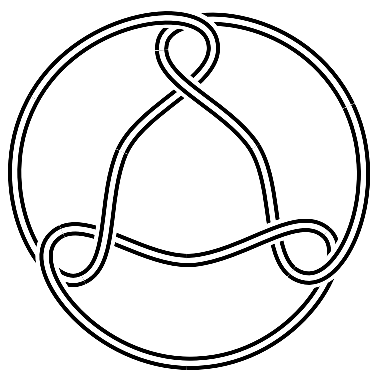
$$|\mathcal{E}(M)| \leq 6.$$

Moreover, any two exceptional slopes have mutual distance no larger than 4 *unless* M *is the Whitehead sister link exterior.*

Magic manifold

The exterior of the following link is called the *magic manifold*. Why?

One gets most of hyperbolic 3-manifolds known and most of the interesting non-hyperbolic Dehn fillings from the magic manifold. Exceptional slopes $= \{-3, -2, -1, 0, 1/0\}$.



Manifold with boundary a union of tori

Define
$\Delta^k(\mathcal{X}_1, \mathcal{X}_2) = \max\{\Delta(\alpha_1, \alpha_2) |$ there is a hyperbolic 3-manifold M such that ∂M is a disjoint union of k tori, and slopes α_1, α_2 on some component of ∂M, such that $M(\alpha_i)$ is of type $\mathcal{X}_i, i = 1, 2\}$.

Δ	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	1	0	2	3
$\mathcal{D}$		1	2	2
$\mathcal{A}$			5	5
$\mathcal{T}$				8

Δ^2	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	1	0	2	2-3
$\mathcal{D}$		1	2	2
$\mathcal{A}$			5	5
$\mathcal{T}$				5

Δ^3	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	0	0	1	1
$\mathcal{D}$		0	1	1
$\mathcal{A}$			3	3
$\mathcal{T}$				3-5

$\Delta^k (k \geq 4)$	$\mathcal{S}$	$\mathcal{D}$	$\mathcal{A}$	$\mathcal{T}$
$\mathcal{S}$	0	0	1	1
$\mathcal{D}$		1	1	1
$\mathcal{A}$			2-3	2-3
$\mathcal{T}$				2-5

Theorem (Lee-Teragaito). *If $k \geq 4$, $\Delta^k(\mathcal{A}, \mathcal{T}) \leq 2$.*

Theorem (Lee). *If $k \geq 4$, $\Delta^k(\mathcal{A}, \mathcal{A}) \leq 2$.*

Theorem (Lee-Teragaito). *If $k \geq 4$, $\Delta^k(\mathcal{T}, \mathcal{T}) \leq 2$.*

Remark. *Any k-component hyperbolic link exterior has at most 4 exceptional slopes ($k \geq 4$).*

$\Delta \leq ?$	0	1	2	3	4	5	6	7	8
$\sharp\{\text{slopes}\} \leq ?$	1	3	4	6	6	8	8	10	12

Dehn surgeries on knots in S^3

Conjecture. Let K be a hyperbolic knot in S^3. Then any exceptional Dehn surgery slope r is either integral, or half-integral and $K(r)$ is toroidal.

$\mathcal{L}(K) = \{r \in \mathcal{E}(K) | K(r)$ is a lens space$\}$
$\mathcal{S}(K) = \{r \in \mathcal{E}(K) | K(r)$ is a small Seifert fibered space$\}$
$\mathcal{T}(K) = \{r \in \mathcal{E}(K) | K(r)$ is toroidal$\}$

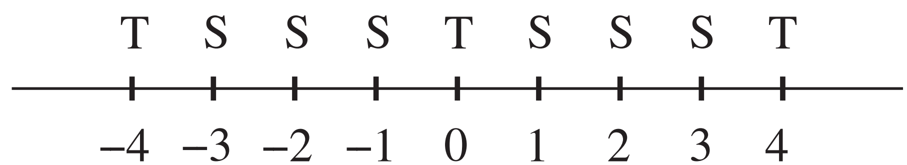
It is conjectured that

$$\mathcal{E}(K) = \mathcal{L}(K) \cup \mathcal{S}(K) \cup \mathcal{T}(K).$$

Examples

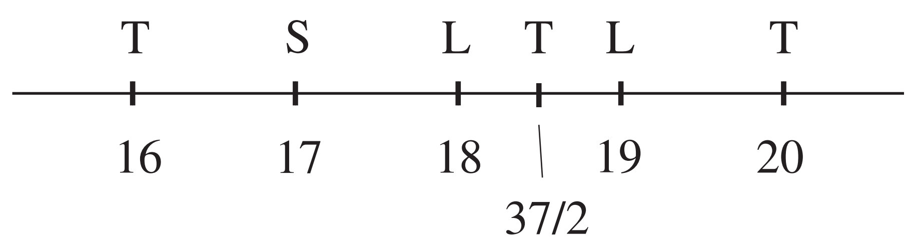
(1) figure-eight knot

$$\mathcal{E}(K) = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \infty\}$$



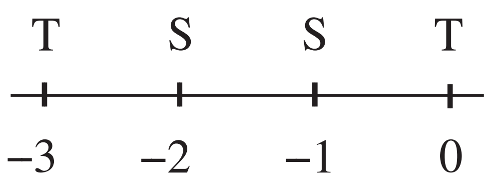
(2) $(-2, 3, 7)$-pretzel knot

$$\mathcal{E}(K) = \{16, 17, 18, 37/2, 19, 20, \infty\}$$

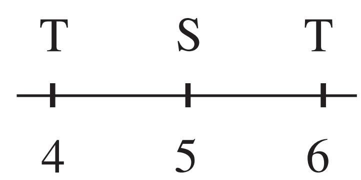


More examples (Ying-Qing Wu)

(−1/2,1/3,2/11)-Montesinos knot



(−1/3,−2/5,2/3)-Montesinos knot



Conjecture. Integral exceptional slopes are consecutive. Moreover, integral toroidal slopes appear at the border, except figure-eight knot.

Theorem. *(Cyclic Surgery Theorem) If a hyperbolic knot has two lens spaces surgery slopes, then they are consecutive.*

Conjecture. If a hyperbolic knot has two lens spaces surgery slopes r and $r+1$, then $\frac{2r+1}{2}$ is a toroidal slope.