

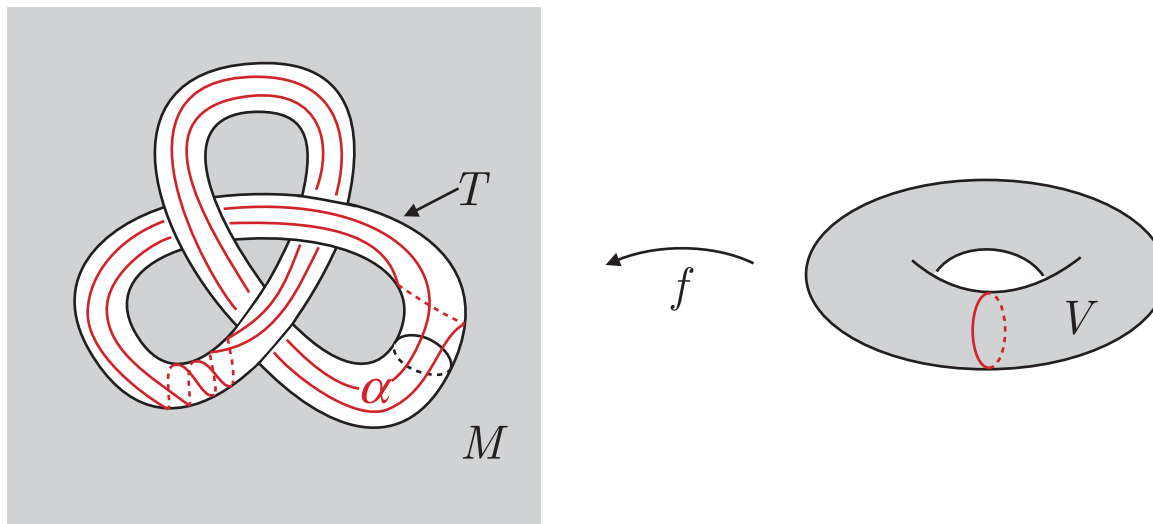
Exceptional Dehn fillings

July 9, 2007

Sangyop Lee

Dehn fillings

M is a compact, connected, orientable 3-manifold with a torus boundary component T .



$$M(\alpha) = M \cup_f V$$

A *slope* is the isotopy class of an essential circle on T ($T \subset \partial M$).

A compact orientable 3-manifold M is *hyperbolic* if M with its boundary tori removed has a finite volume complete hyperbolic structure.

Theorem (Hyperbolic Dehn Surgery Theorem). *If M is a hyperbolic 3-manifold with a torus boundary component T , then $M(\alpha)$ are hyperbolic for all but finitely many slopes α on T .*

Research Aim

M : hyperbolic \longrightarrow $M(\alpha)$: not hyperbolic for finitely many slopes

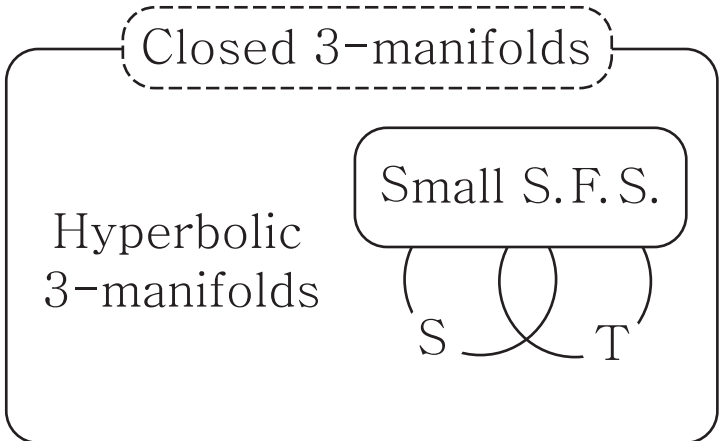
Such slopes are called *exceptional slopes*.

Project How many exceptional slopes?

Example The figure-8 knot exterior has 10 exceptional slopes.

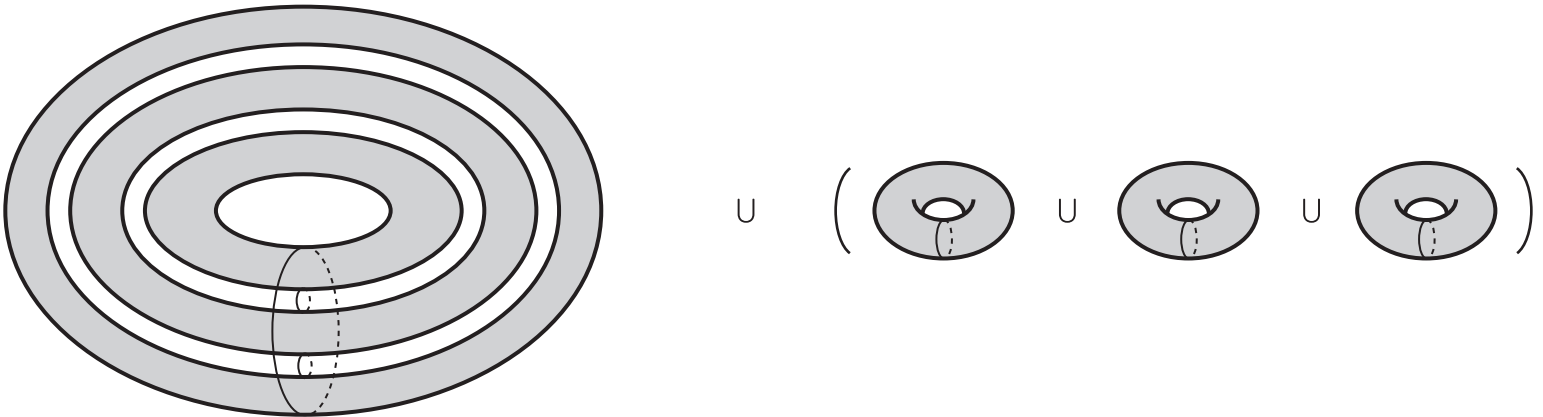
Conjecture (Gordon). There are at most 8 exceptional slopes for hyperbolic 3-manifolds except the figure-8 knot exterior.

Geometrization Conjecture A closed 3-manifold is not hyperbolic if and only if it is reducible, toroidal, or a small Seifert fibered space.



Small Seifert fibered spaces

Small Seifert fibered spaces are obtained from $P \times S^1$ by suitably performing Dehn filling 3 times, where P is a pair of pants.

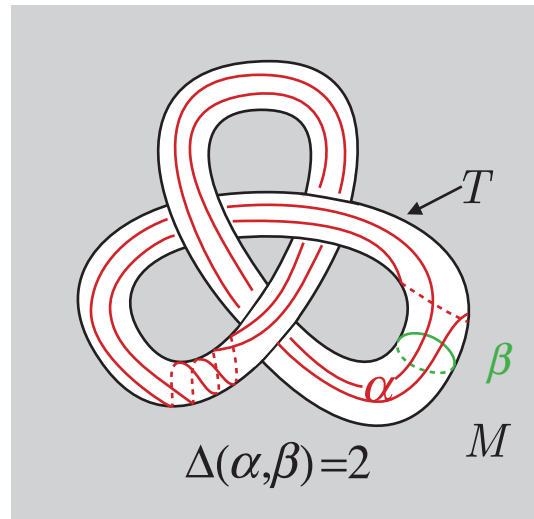


Distance between two slopes

Let α, β be two slopes on $T \subset \partial M$.

$\Delta(\alpha, \beta) :=$ minimal geometric intersection number of α and β .

Example



Parameterizing slopes

K : a knot in S^3 , $M_K = S^3 - \text{int}N(K)$

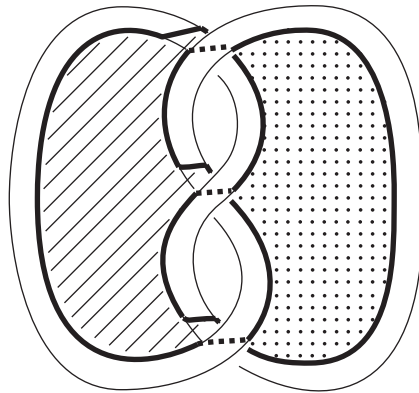
μ, λ : meridian and longitude $\subset \partial M_K$.

α : an essential simple closed curve on ∂M_K

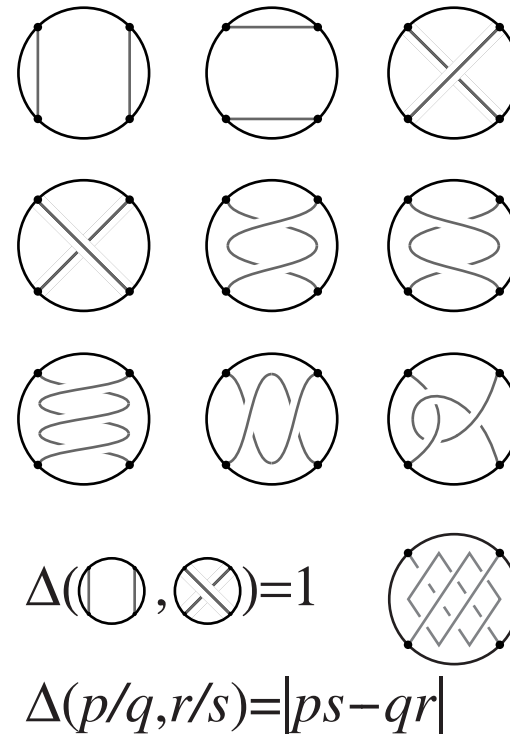
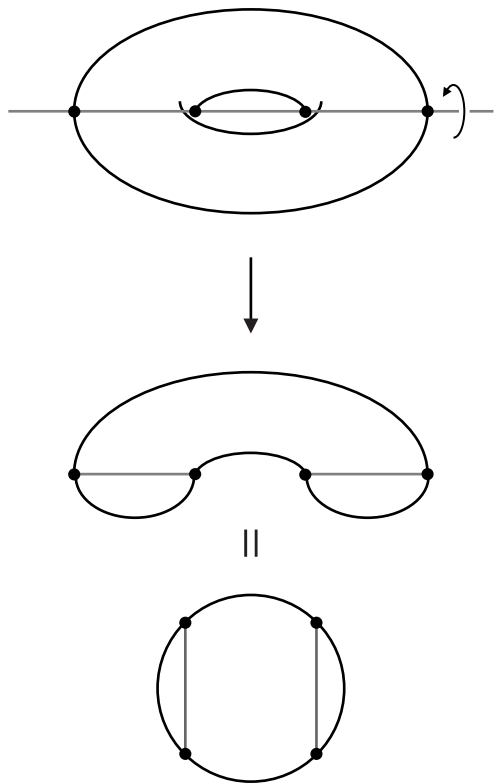
$\alpha \sim m\mu + l\lambda$ for some coprime integers m, l .

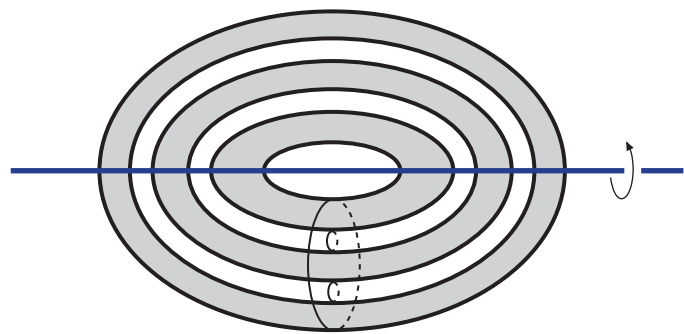
$\{\text{Slopes}\} \leftrightarrow \mathbb{Q} \cup \{1/0\}$, $\alpha \leftrightarrow m/l$

$\Delta(p/q\text{-slope}, r/s\text{-slope}) = |ps - qr|$

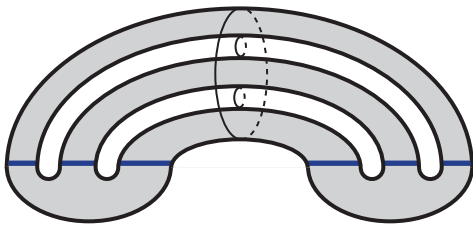
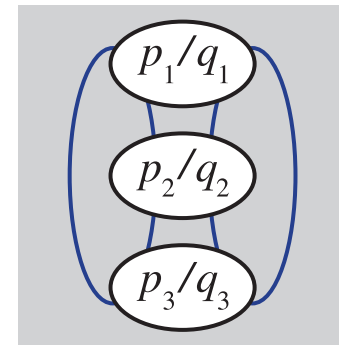


Double branched coverings and rational tangles

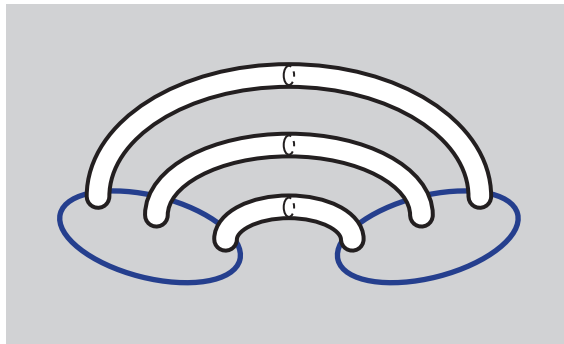




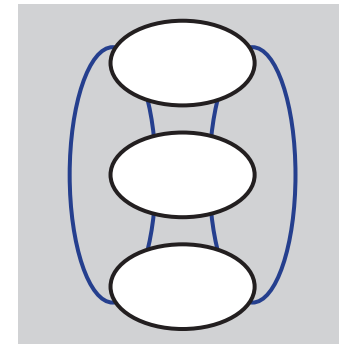
Small Seifert
Fibered Space



~



~

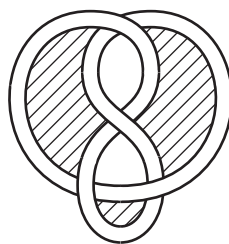
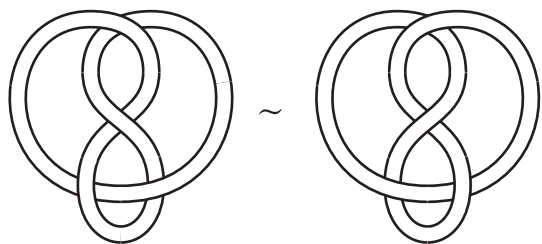


The figure-8 knot exterior and exceptional slopes

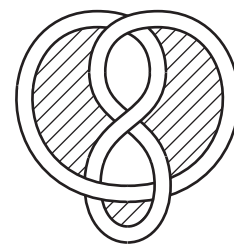
Let M be the exterior of the figure-8 knot.

Then $\mathcal{E}(M) = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 1/0\}$

Since the figure-8 knot is amphicheiral, $M(r) \cong M(-r)$.

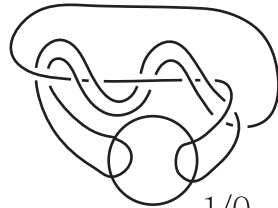
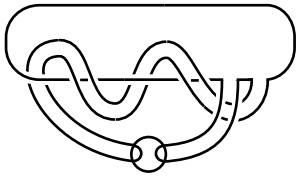
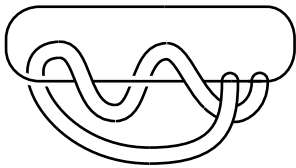
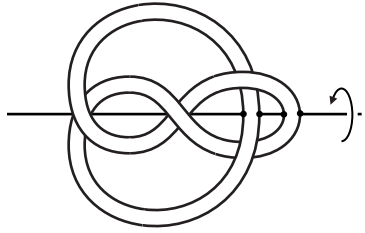


boundary slope 4

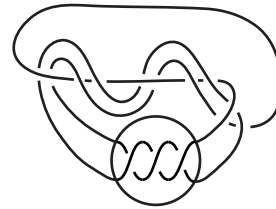
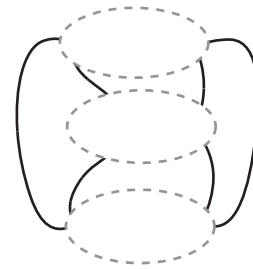


boundary slope -4

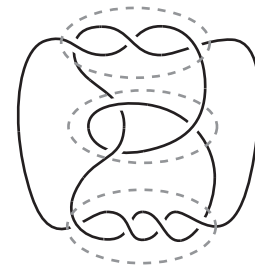
$\Delta(4, -4) = 8$



1/0



3



A closed 3-manifold is *hyperbolike* if it is irreducible, atoroidal, and is not a small Seifert fibered space.

Let M be a hyperbolic 3-manifold with a torus boundary component T . Define

$$\mathcal{E}(M; T) = \mathcal{E}(M) = \{ \alpha \subset T \mid M(\alpha) \text{ is not hyperbolic} \}$$

$$\mathcal{E}'(M; T) = \mathcal{E}'(M) = \{ \alpha \subset T \mid M(\alpha) \text{ is not hyperbolike} \} \subset \mathcal{E}(M).$$

Then Gordon's conjecture is reformulated as follows.

Conjecture $|\mathcal{E}(M)| \leq 8$ if M is not the figure-8 knot exterior.

Known results

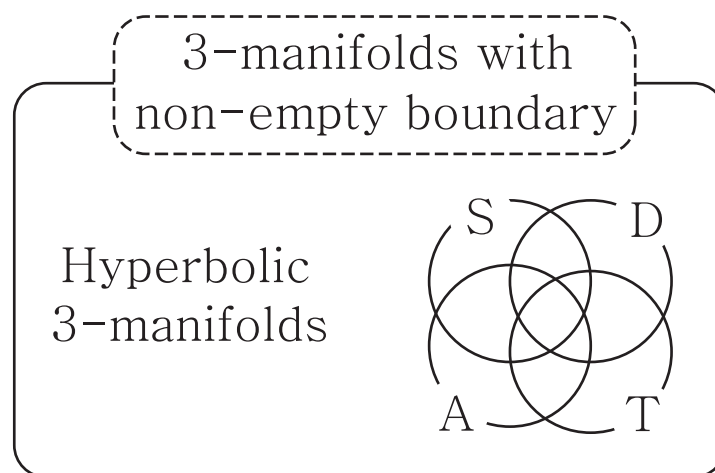
Theorem (Bleiler-Hodgson). *Let M be a hyperbolic 3-manifold with ∂M a single torus. Then $|\mathcal{E}'(M)| \leq 24$.*

Agol and Lackenby greatly improved this estimation.

Theorem (Agol, Lackenby). $|\mathcal{E}'(M)| \leq 12$.

What if ∂M is not a single torus?

Theorem (Geometrization Theorem for Haken manifolds). *A compact 3-manifold with non-empty boundary is not hyperbolic if and only if it is reducible (S), boundary-reducible (\mathcal{D}), annular (\mathcal{A}), or toroidal (\mathcal{T}).*



Known results (continued)

$\Delta \leq ?$	\mathcal{S}	\mathcal{D}	\mathcal{A}	\mathcal{T}
\mathcal{S}	1	0	2	3
\mathcal{D}		1	2	2
\mathcal{A}			5	5
\mathcal{T}				8

Upper bounds for Δ

For example, $\Delta(\mathcal{S}, \mathcal{A}) \leq 2$ means:

Given a hyperbolic manifold M , if $M(\alpha), M(\beta)$ each contain an essential sphere and an essential annulus, then $\Delta(\alpha, \beta) \leq 2$ [Wu, Qiu].

$\Delta \leq ?$	0	1	2	3	4	5	6	7	8
$\#\{\text{slopes}\} \leq ?$	1	3	4	6	6	8	8	10	12

Theorem (Gordon). *Let M be a hyperbolic 3-manifold such that both $M(\alpha)$ and $M(\beta)$ contain an essential torus. Then either*

(1) $\Delta(\alpha, \beta) \leq 5$; or

(2) $\Delta(\alpha, \beta) = 6$ and $M \cong W(-2)$; or

(3) $\Delta(\alpha, \beta) = 7$ and $M \cong W(5/2)$; or

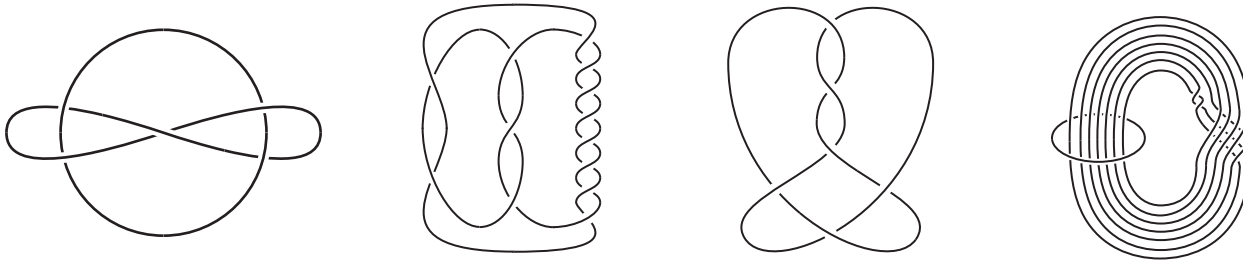
(4) $\Delta(\alpha, \beta) = 8$ and $M \cong W(-1)$ or $M \cong W(5)$.

Therefore in general, $|\mathcal{E}(M)| \leq 8$ for hyperbolic 3-manifolds M with at least two boundary components.

Examples

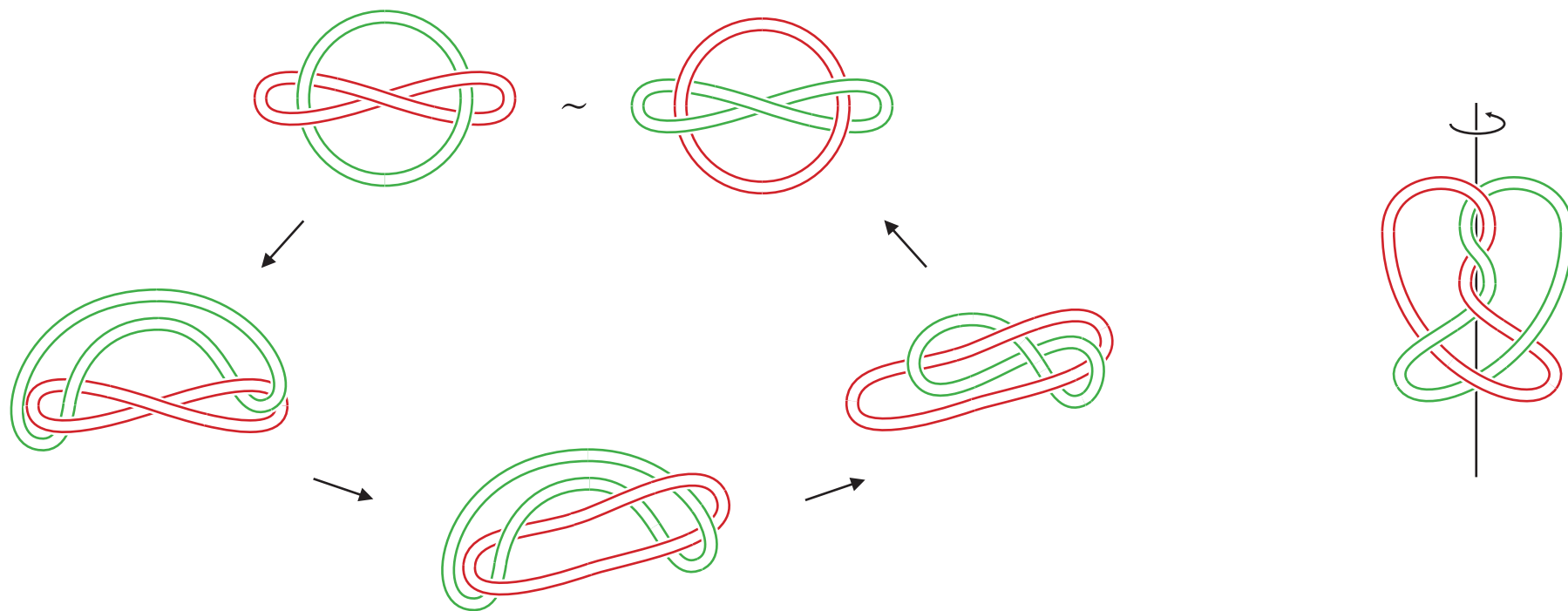
For hyperbolic 3-manifolds M with at least two boundary components, the maximal observed value for $|\mathcal{E}(M)|$ is 6.

The following links are the Whitehead link, the Whitehead sister link, the 2-bridge link associated to $3/10$ in Conway's notation, and the Berge link.

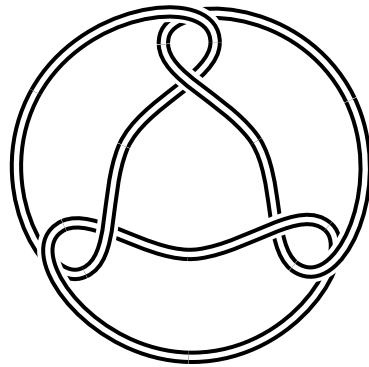


Theorem (Martelli-Petronio). *Their exteriors have exactly 6 exceptional slopes.*

Let M_1, M_2, M_3, M_4 be the exteriors of the above links, respectively. Then there is a self-homeomorphism of M_i interchanging two boundary tori, $i = 1, 2, 3, 4$. For M_1 and M_3 , see the following figure.



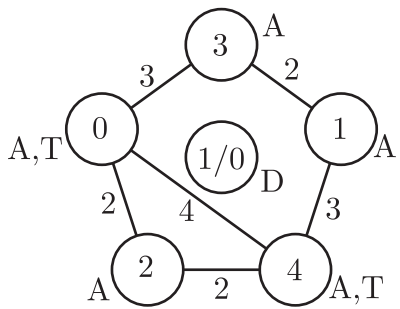
In fact, these manifolds are obtained from the exterior of the below link by Dehn filling.



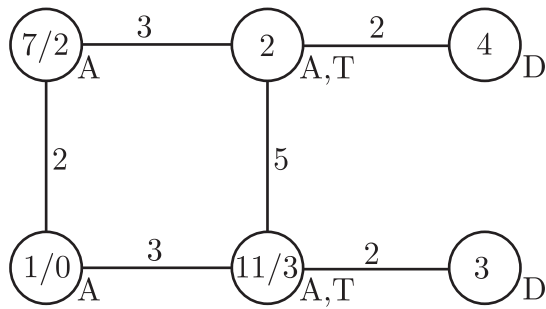
Hence, there is the desired homeomorphism for each M_i .

Exceptional slopes for the link exteriors

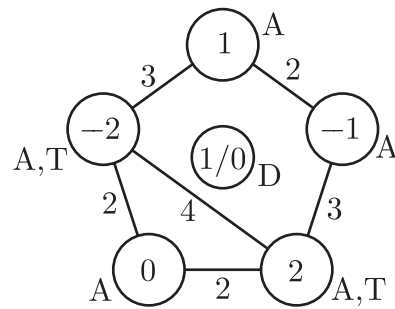
The exceptional slopes for M_1, M_2, M_3, M_4 and the mutual distances are shown below. (Unlabelled distances are 1.)



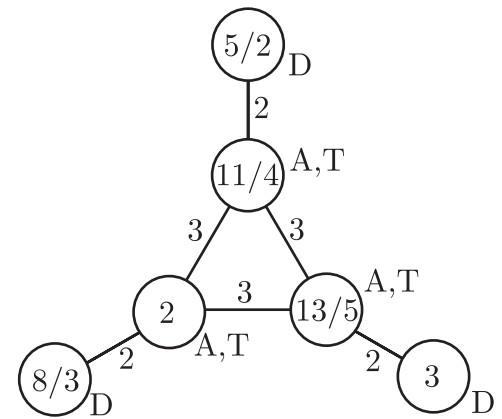
exceptional slopes for M_1



exceptional slopes for M_2



exceptional slopes for M_3



exceptional slopes for M_4

Theorem (Frigerio-Martelli-Petronio). *For any $g \geq 2$, there are infinitely many hyperbolic 3-manifolds with 6 exceptional Dehn fillings among which three yield a handlebody of genus g and the others yield an annular manifold.*

We remark that these manifolds are built by generalizing the Berge link exterior.

Theorem (Lee). *Let M be a hyperbolic 3-manifold with one torus boundary component and at least one other boundary component. Then*

$$|\mathcal{E}(M)| \leq 6.$$

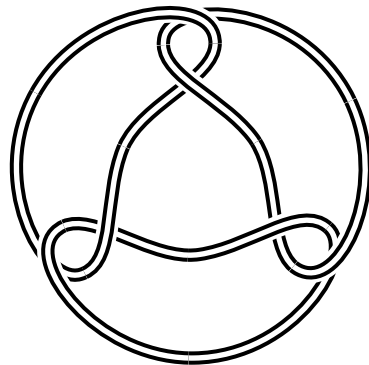
Moreover, any two exceptional slopes have mutual distance no larger than 4 unless M is the Whitehead sister link exterior.

Magic manifold

The exterior of the following link is called the *magic manifold*. Why?

One gets most of hyperbolic 3-manifolds known and most of the interesting non-hyperbolic Dehn fillings from the magic manifold.

Exceptional slopes = $\{-3, -2, -1, 0, 1/0\}$.



Manifold with boundary a union of tori

Define

$\Delta^k(\mathcal{X}_1, \mathcal{X}_2) = \max\{\Delta(\alpha_1, \alpha_2) \mid \text{there is a hyperbolic 3-manifold } M \text{ such that } \partial M \text{ is a disjoint union of } k \text{ tori, and slopes } \alpha_1, \alpha_2 \text{ on some component of } \partial M, \text{ such that } M(\alpha_i) \text{ is of type } \mathcal{X}_i, i = 1, 2\}$.

Δ	\mathcal{S}	\mathcal{D}	\mathcal{A}	\mathcal{T}
\mathcal{S}	1	0	2	3
\mathcal{D}		1	2	2
\mathcal{A}			5	5
\mathcal{T}				8

Δ^2	\mathcal{S}	\mathcal{D}	\mathcal{A}	\mathcal{T}
\mathcal{S}	1	0	2	2-3
\mathcal{D}		1	2	2
\mathcal{A}			5	5
\mathcal{T}				5

Δ^3	\mathcal{S}	\mathcal{D}	\mathcal{A}	\mathcal{T}
\mathcal{S}	0	0	1	1
\mathcal{D}		0	1	1
\mathcal{A}			3	3
\mathcal{T}				3-5

$\Delta^k(k \geq 4)$	\mathcal{S}	\mathcal{D}	\mathcal{A}	\mathcal{T}
\mathcal{S}	0	0	1	1
\mathcal{D}		1	1	1
\mathcal{A}			2-3	2-3
\mathcal{T}				2-5

Theorem (Lee-Teragaito). *If $k \geq 4$, $\Delta^k(\mathcal{A}, \mathcal{T}) \leq 2$.*

Theorem (Lee). *If $k \geq 4$, $\Delta^k(\mathcal{A}, \mathcal{A}) \leq 2$.*

Theorem (Lee-Teragaito). *If $k \geq 4$, $\Delta^k(\mathcal{T}, \mathcal{T}) \leq 2$.*

Remark. *Any k -component hyperbolic link exterior has at most 4 exceptional slopes ($k \geq 4$).*

$\Delta \leq ?$	0	1	2	3	4	5	6	7	8
$\#\{\text{slopes}\} \leq ?$	1	3	4	6	6	8	8	10	12

Dehn surgeries on knots in S^3

Conjecture. Let K be a hyperbolic knot in S^3 . Then any exceptional Dehn surgery slope r is either integral, or half-integral and $K(r)$ is toroidal.

$$\mathcal{L}(K) = \{r \in \mathcal{E}(K) \mid K(r) \text{ is a lens space}\}$$

$$\mathcal{S}(K) = \{r \in \mathcal{E}(K) \mid K(r) \text{ is a small Seifert fibered space}\}$$

$$\mathcal{T}(K) = \{r \in \mathcal{E}(K) \mid K(r) \text{ is toroidal}\}$$

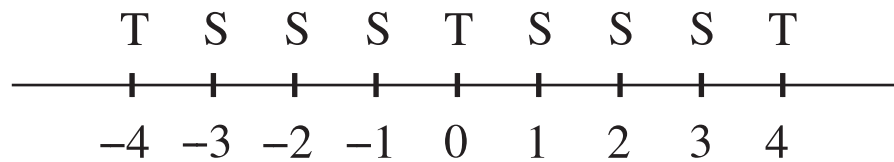
It is conjectured that

$$\mathcal{E}(K) = \mathcal{L}(K) \cup \mathcal{S}(K) \cup \mathcal{T}(K).$$

Examples

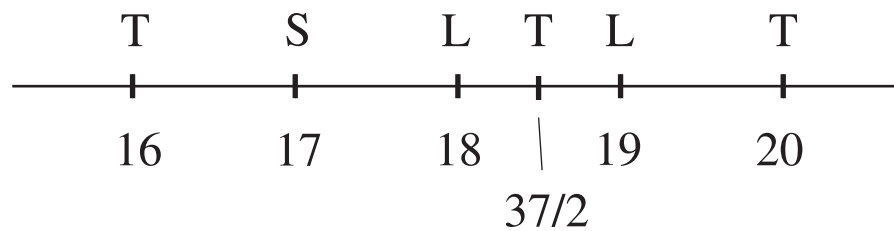
(1) figure-eight knot

$$\mathcal{E}(K) = \{0, \pm 1, \pm 2, \pm 3, \pm 4, \infty\}$$



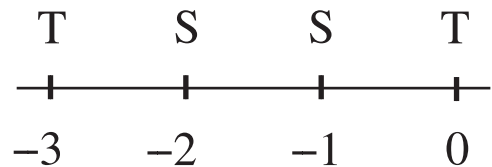
(2) $(-2, 3, 7)$ -pretzel knot

$$\mathcal{E}(K) = \{16, 17, 18, 37/2, 19, 20, \infty\}$$

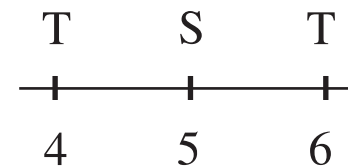


More examples (Ying-Qing Wu)

$(-1/2, 1/3, 2/11)$ -Montesinos knot



$(-1/3, -2/5, 2/3)$ -Montesinos knot



Conjecture. Integral exceptional slopes are consecutive. Moreover, integral toroidal slopes appear at the border, except figure-eight knot.

Theorem. (*Cyclic Surgery Theorem*) *If a hyperbolic knot has two lens spaces surgery slopes, then they are consecutive.*

Conjecture. If a hyperbolic knot has two lens spaces surgery slopes r and $r + 1$, then $\frac{2r+1}{2}$ is a toroidal slope.

$$\begin{array}{ccc} \text{L} & \text{T} & \text{L} \\ \hline + & + & + \\ r & \frac{2r+1}{2} & r+1 \end{array}$$