PERGAMON

Topology 39 (2000) 549–556

TOPOLOGY

www.elsevier.com/locate/top

Spaces with nonpositive curvature and their ideal boundaries

Christopher B. Croke[a, 1], Bruce Kleiner[b, *, 2]

[a]David Rittenhouse Laboratory, Department of Mathematics, University of Pennsylvania, 209 S. 33rd St., Philadelphia, PA 19104–6395, USA

[b]Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090, USA

Received 18 February 1998; received in revised form 10 December 1998; accepted 11 December 1998

Abstract

We construct a pair of finite piecewise Euclidean 2-complexes with nonpositive curvature which are homeomorphic but whose universal covers have nonhomeomorphic ideal boundaries, settling a question of Gromov. © 2000 Elsevier Science Ltd. All rights reserved.

1. Introduction

The ideal boundary of a locally compact Hadamard space[3] X is a compact metrizable space on which the isometry group of X acts by homeomorphisms. Even though the ideal boundary is a well-known construct with many applications in the literature (see for example [10, 4, 2]), the action of the isometry group on the boundary has not been studied closely except in the case of symmetric spaces, Gromov hyperbolic spaces, Euclidean buildings, and a handful of other cases. In the Gromov hyperbolic case[4] the boundary behaves nicely with respect to quasi-isometries: any quasi-isometry $f: X_1 \to X_2$ between Gromov hyperbolic Hadamard spaces induces a boundary homeomorphism $\partial_\infty f: \partial_\infty X_1 \to \partial_\infty X_2$ [7]. This has the consequence that the ideal boundary is "geometry independent":

If a finitely generated group G acts discretely, cocompactly and isometrically on two Gromov hyperbolic Hadamard spaces X_1, X_2, then there is a G-equivariant homeomorphism $\partial_\infty X_1 \to \partial_\infty X_2$.

* Corresponding author. Tel.: +1-801-583-8594; fax: +1-801-581-4148; e-mail: bkleiner@math.utah.edu.

[1] Supported by NSF grants DMS-95-05175 and DMS-96-26-232.

[2] Supported by a Sloan Foundation Fellowship, and NSF grants DMS-95-05175, DMS-96-26911, DMS-9022140.

[3] Following [3] we will refer to complete, simply connected length spaces with nonpositive curvature as Hadamard spaces.

[4] The same statement is true of higher rank irreducible symmetric spaces and Euclidean buildings by [9].

PII: S0040-9383(99)00016-6

In [8, p. 136] Gromov asked whether this fundamental property still holds if the hyperbolicity assumption is dropped. Buyalo [5] and the authors [6] independently answered Gromov's question negatively: [5, 6] exhibit a pair of deck group invariant Riemannian metrics on a universal cover which have ideal boundaries homeomorphic to S^2, such that the deck group actions on the boundaries are topologically inequivalent. Gromov also asked if $\partial_\infty X_1$ must be (non-equivariantly) homeomorphic to $\partial_\infty X_2$ whenever X_1 and X_2 are Hadamard spaces admitting discrete, cocompact, isometric actions by the same finitely generated group G. In this paper we show that even this can fail:

Theorem 1. *There is a pair $\bar{X}_1, \bar{X}_2$ of homeomorphic finite 2-complexes with nonpositive curvature such that the universal covers X_1, X_2 have nonhomeomorphic ideal boundaries.*

We remark that if M_1 and M_2 are closed Riemannian manifolds with nonpositive curvature and $\pi_1(M_1) \simeq \pi_1(M_2)$, then their universal covers will have ideal boundaries homeomorphic to spheres of the same dimension.

Although some basic questions about the boundary have now been answered, a number of related issues are wide open, except in a few special cases. It would be interesting to know exactly which geometric features determine the ideal boundary of a Hadamard space up to (equivariant) homeomorphism. This question has a clean answer (see [6]) in the case of graph manifolds or the 2-complexes considered in this paper. In order to answer the question in any generality, it appears that it will be necessary to develop a kind of "generalized symbolic dynamics" for geodesic flows of nonpositively curved spaces.

2. Notation and preliminaries

A reference for the facts recalled here is [3]. If X is a Hadamard space, then we denote the ideal boundary of X by $\partial_\infty X$, the geodesic segment joining $x_1, x_2 \in X$ by $\overline{x_1 x_2}$, and the geodesic ray leaving $p \in X$ in the asymptote class of $\xi \in \partial_\infty X$ by $\overline{p\xi}$. If $p \in X$, $\xi_1, \xi_2 \in \partial_\infty X$, then $\angle_p(\xi_1, \xi_2)$ is the angle between the initial velocities of the rays $\overline{p\xi_1}, \overline{p\xi_1}$. $\angle_T(\xi_1, \xi_2) := \sup_{p \in X} \angle_p(\xi_1, \xi_2)$ will denote the Tits angle between $\xi_1, \xi_2 \in \partial_\infty X$. If $p \in X$ then $\angle_p(\xi_1, \xi_2) = \angle_T(\xi_1, \xi_2)$ iff the rays $\overline{p\xi_1}$ and $\overline{p\xi_2}$ bound a flat sector.

By the Cartan–Hadamard theorem [1, 3], the universal cover of a connected, complete, length space with nonpositive curvature is a Hadamard space with the natural metric. Let Z be a complete, connected space with nonpositive curvature, and let $\pi : \tilde{Z} \to Z$ be the universal cover. If $Y \subset Z$ is a closed, connected, locally convex subset, then the induced length metric on Y has nonpositive curvature, $\pi^{-1}(Y) \subset \tilde{Z}$ is a disjoint union of closed convex components isometric to $\tilde{Y}$, and the induced homomorphism $\pi_1(Y) \to \pi_1(Z)$ is a monomorphism.

3. Torus complexes

The following piecewise Euclidean 2-complexes were suggested to us by Bernhard Leeb, after a discussion of the graph manifold geometry in [6].

Let T_0, T_1, T_2 be flat two-dimensional tori. For $i = 1, 2$, we assume that there are (primitive) closed geodesics $a_i \subset T_0$ and $b_i \subset T_i$ with $length(a_i) = length(b_i)$, and we glue T_i to T_0 by identifying a_i with b_i isometrically. We assume that a_1 and a_2 lie in distinct free homotopy classes, and intersect once at an angle $\alpha \in (0, \pi/2]$. The resulting 2-complex $\bar{X}$ is nonpositively curved as a length space because gluing of nonpositively curved spaces along locally convex subsets produces a nonpositively curved space [3]. We refer to $\bar{X}$ as a *torus complex*. For $i = 1, 2$ let $\bar{Y}_i := T_0 \cup T_i \subset \bar{X}$. Notice that $\bar{Y}_i$ and T_0 are closed, locally convex subsets of $\bar{X}$. Therefore, the inclusions $\bar{Y}_i \subset \bar{X}$ and $T_0 \subset \bar{X}$ induce monomorphisms of fundamental groups.

4. The structure of the universal cover

Let $\pi : X \to \bar{X}$ be the universal covering of $\bar{X}$. X is a Hadamard space by the Cartan–Hadamard theorem. A *block* is a connected component of $\pi^{-1}(\bar{Y}_i) \subset X$, and a *wall* is a connected component of $\pi^{-1}(T_0) \subset X$. Let $\mathscr{B}$ and $\mathscr{W}$ denote the (locally finite) collections of blocks and walls in X. Each block (resp. wall) is a closed, connected, locally convex subset of X. Hence by Section 2 each block (resp. wall) is a convex subset of X which is intrinsically isometric to the universal cover of $\bar{Y}_i$ (resp. T_0). If $W \in \mathscr{W}$, $B \in \mathscr{B}$, then either $W \cap B = \emptyset$ or $W \cap B = W$ since $W \cap B$ is open and closed in W; W is contained in precisely two blocks, one covering $\bar{Y}_1$ and the other covering $\bar{Y}_2$. If $B_1, B_2 \in \mathscr{B}$ are distinct blocks and $B_1 \cap B_2 \neq \emptyset$, then (after relabelling if necessary) B_i covers $\bar{Y}_i$ and so $B_1 \cap B_2$ consists of a (convex) union of walls; therefore $B_1 \cap B_2 = W$ for some $W \in \mathscr{W}$. When $B_1 \cap B_2 \neq \emptyset$ we will say that the blocks B_1 and B_2 are adjacent.

$\bar{Y}_i$ is a "flat" S^1 bundle over a bouquet of two circles, so the universal cover Y_i of $\bar{Y}_i$ (and hence each block) is isometric to the metric product of a simplicial tree with R. A *singular geodesic of a block B* is the inverse image of a vertex under the projection of B to its tree factor. Note that singular geodesics of adjacent blocks which lie in the common wall intersect at angle α.

The nerve of $\mathscr{B}$ (the simplicial complex recording (multiple) intersections of blocks) is a simplicial tree. (This is just the Bass–Serre tree of the amalgamated free product decomposition $\pi_1(\bar{X}) = \pi_1(\bar{Y}_1) *_{\pi_1(T_0)} \pi_1(\bar{Y}_2)$.) To see this note that if $\varepsilon > 0$ is sufficiently small and $\mathscr{B}_\varepsilon$ is the collection of (open) ε-tubular neighborhoods of blocks, then $Nerve(\mathscr{B}_\varepsilon)$ is isomorphic to $Nerve(\mathscr{B})$. Using a partition of unity subordinate to this cover of $|Nerve(\mathscr{B}_\varepsilon)|$ one gets a continuous map ϕ: $X \to |Nerve(\mathscr{B}_\varepsilon)|$. Any map $\gamma : S^1 \to |Nerve(\mathscr{B})|$ can be "lifted" to X up to homotopy: there is a $\hat{\gamma} : S^1 \to X$ so that $p \circ \hat{\gamma}$ is homotopic to γ. Since $\pi_1(X)$ is trivial, this implies that $\pi_1(|Nerve(\mathscr{B})|)$ is trivial. In particular, every wall separates X. We will say that a wall (resp. block) separates two blocks $B_1, B_2 \in \mathscr{B}$ if the edge (resp. vertex) of $|Nerve(\mathscr{B})|$ corresponding to the wall (resp. vertex) lies between the vertices of $|Nerve(\mathscr{B})|$ corresponding to B_1 and B_2.

Our plan is to show that the subspace $\bigcup_{B \in \mathscr{B}} \partial_\infty B \subset \partial_\infty X$ can be characterized purely topologically[5], and that its topology is different depending on whether $\alpha = \pi/2$ or not. It will then follow that a torus complex with $\alpha < \pi/2$ and a torus complex with $\alpha = \pi/2$ have universal covers with nonhomeomorphic ideal boundaries.

[5] At first glance one might think that $\bigcup_{B \in \mathscr{B}} \partial_\infty B$ is a path component of $\partial_\infty X$, but this turns out not to be the case. It is a "safe" path component, see Section 7.

5. Itineraries

For each $p \in X \backslash \bigcup_{W \in \mathscr{W}} W$, $\xi \in \partial_\infty X$, we get a sequence of blocks B_i called the *p-itinerary* (simply the *itinerary* if the basepoint p is understood) of ξ, as follows. Let B_i be the ith block that the ray $\overline{p\xi}$ enters; the ray *enters* a block B if it reaches a point in $B \backslash \bigcup_{W \in \mathscr{W}} W$. We will denote the p-itinerary of $\overline{p\xi}$ by $Itin(\overline{p\xi})$ or $Itin(\xi)$.

Lemma 2. *The itinerary of any $\xi \in \partial_\infty X$ is the sequence of successive vertices of a geodesic segment or geodesic ray in the simplical tree $Nerve(\mathscr{B})$.*

Proof. Blocks are convex, so a geodesic cannot revisit any block which it left. The topological frontier of any $B \in \mathscr{B}$ is the union of the walls contained in B, so a geodesic segment which leaves B must arrive at a wall $W \subset B$, and then enter the block $B' \in \mathscr{B}$ adjacent to B along W. The collection $\mathscr{B}$ is locally finite, so the lemma follows. □

Note that $\xi \in \partial_\infty X$ has a finite itinerary iff $\xi \in \partial_\infty B$ for some $B \in \mathscr{B}$.

6. Local components of $\partial_\infty X$

Since each block B is isometric to the product of simplicial tree with R, $\partial_\infty B$ is homeomorphic to the suspension of a Cantor set. A *pole* of B is one of the two suspension points in $\partial_\infty B$.

Lemma 3. *If $B_1, B_2 \in \mathscr{B}$, then one of the following holds*:

1. $\partial_\infty B_1 \cap \partial_\infty B_2 = \emptyset$.
2. $B_1 \cap B_2 = W \in \mathscr{W}$ and $\partial_\infty B_1 \cap \partial_\infty B_2 = \partial_\infty W$.
3. *There is a $B \in \mathscr{B}$ such that $B \cap B_i = W_i \in \mathscr{W}$ and $\partial_\infty B_1 \cap \partial_\infty B_2$ is the set of poles of B.*

Proof. Suppose $B_1, B_2 \in \mathscr{B}$ are distinct blocks, $\xi \in \partial_\infty B_1 \cap \partial_\infty B_2$, and $W \in \mathscr{W}$ is a wall separating B_1 from B_2. Choose basepoints $b_i \in B_i$, $w \in W$. If $x_k \in \overline{b_1\xi}$ is a sequence tending to infinity, and $y_k \in \overline{b_2\xi}$ is a sequence with $d(y_k, x_k) < C$, then we can find a $z_k \in \overline{x_k y_k} \cap W$ since W separates B_1 from B_2. Therefore, $\overline{wz_k} \subset W$ converges, and the limit ray $\overline{w\xi}$ lies in W. Hence $\xi \in \partial_\infty W$.

Note that if $W_1, W_2 \subset B \in \mathscr{B}$, then $\partial_\infty W_1 \cap \partial_\infty W_2$ is just the set of poles of B; and $\xi \in \partial_\infty X$ cannot be a pole of two adjacent blocks simultaneously.

The lemma follows, since $\partial_\infty B_1 \cap \partial_\infty B_2 \neq \emptyset$ now implies that the combinatorial distance between B_1 and B_2 in $Nerve(\mathscr{B})$ is $\leqslant 2$. □

Lemma 4. *Suppose ξ lies on the ideal boundary of a block $B \in \mathscr{B}$, and assume ξ is not a pole of any block other than B. Then the path component of ξ in a suitable neighborhood Ω of ξ is contained in $\partial_\infty B$.*

Proof. *Case* I: $\xi \in \partial_\infty B$ is a pole of B. Choose $p \in B \backslash \bigcup_{W \in \mathscr{W}} W$. Recall (see Section 3) that α is the angle between singular geodesics of adjacent blocks lying in the common wall, so α is the minimum Tits angle between ξ and any pole of a block adjacent to B. Let $\Omega := \{\xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \alpha/2\}$, where $\angle_p(\xi, \xi')$ is the angle between the initial velocities of the two rays $\overline{p\xi}, \overline{p\xi'}$. We define an *exit from* B to be a singular geodesic $E \subset B$ of a block adjacent to B. A ray $\overline{p\xi'}$ *exits from* B *via* E if $\overline{p\xi'} \cap B$ is a geodesic segment ending at E, and the ray $\overline{p\xi'}$ continues into the block containing E. For each exit E from B, let Ω_E be the set of $\xi' \in \Omega$ such that $\overline{p\xi'}$ exits B via E.

Sublemma 5. *Ω_E is an open and closed subset of Ω.*

Proof. *Openness*: If $\xi' \in \Omega_E$, then $\overline{p\xi'} \cap B$ is a segment ending at some $e \in E$, and $\overline{p\xi'}$ enters the block B' adjacent to B which contains E. But then any sufficiently nearby (in the cone topology) ray $\overline{p\xi''}$ also leaves B at a point close to e; clearly this point must lie on E as the collection of exits is discrete. Therefore Ω_E is open in $\partial_\infty X$.

Closedness: Let $E' \subset E$ be the set of "exit points" for elements of Ω_E: the endpoints of segments $\overline{p\xi'} \cap B$, where $\xi' \in \Omega_E$. E' is bounded, for otherwise we could find a sequence $e_k \in E'$ with $\lim_{k\to\infty} d(e_k, p) = \infty$, and get a limit ray $\overline{pe_\infty} \subset B$ with $e_\infty \in \partial_\infty E \subset \partial_\infty B \cap \partial_\infty B'$, and $\angle_p(\xi, e_\infty) \leqslant \alpha/2$; this is absurd since e_∞ is a pole of B' and so $\angle_p(e_\infty, \xi) = \angle_T(e_\infty, \xi) \geqslant \alpha$. Now suppose $\xi'_k \in \Omega_\mathrm{E}$ and $\lim_{k\to\infty} \xi'_k = \xi'_\infty \in \Omega$. We have, after passing to a subsequence if necessary, that $\overline{p\xi_k'} \cap B = \overline{pe_k}$ where $e_k \in E$ and $\lim_{k\to\infty} e_k = e_\infty \in E$. Then $\overline{p\xi'_\infty} \cap B$ contains $\overline{pe_\infty}$; if $\overline{p\xi'_\infty} \cap B \neq \overline{pe_\infty}$ then clearly $\overline{p\xi'_\infty}$ contains a segment of E, forcing $\overline{pe_\infty} \subset E$, which contradicts the choice of p. Thus we have $\xi'_\infty \in \Omega_E$. □

It follows that the connected (or path) component of ξ in Ω is contained in $\partial_\infty B$, since any subset $C \subseteq \Omega$ containing ξ and intersecting Ω_E admits a separation $C = (C \cap \Omega_E) \cup (C \backslash \Omega_E)$ into open subsets of C, and any $\xi' \in \Omega \backslash \partial_\infty B$ lies in Ω_E for some E.

Case II: *$\xi \in \partial_\infty W$ where W is the wall separating two adjacent blocks B_1, B_2, and ξ is not a pole.* Pick $p \in W$ not lying on a singular geodesic. Let ψ be the minimum Tits distance between ξ and a pole of B_i, $i = 1, 2$, and set

$$\Omega := \left\{\xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \frac{\psi}{2}\right\}.$$

Let E be a singular geodesic of B_1 or B_2 which is contained in W. We say that the ray $\overline{p\xi'}$ *exits W via E* if $\overline{p\xi'} \cap W$ ends at a point in E, and $\overline{p\xi'}$ then immediately enters the block corresponding to E. Let Ω_E be the set of $\xi' \in \Omega$ so that $\overline{p\xi'}$ exits W via E. One checks as in case I that Ω_E is closed and open in Ω, so we conclude that the connected component of ξ in Ω is contained in $\partial_\infty W$.

Case III: *$\xi \in \partial_\infty B$ does not lie in the boundary of any block other than B.* Let ϕ be the minimum Tits angle between ξ and a pole of B, and set

$$\Omega := \left\{\xi' \in \partial_\infty X \mid \angle_p(\xi', \xi) < \frac{\phi}{2}\right\}.$$

Pick $p \in B \setminus \bigcup_{W \in \mathscr{W}} W$. Since ξ is not a pole of B, the ray $\overline{p\xi}$ determines an isometrically embedded Euclidean half-plane $H \subset B$, the intersection of the flat planes in B containing it. Let $\mathscr{B}'$ be the collection of blocks adjacent to B. If $B' \in \mathscr{B}'$ then $B' \cap H$ ($= W \cap H$ where $W = B \cap B'$ is the wall between B and B') is either empty, a singular geodesic of B, or a flat strip with finite width bounded by singular geodesics, for otherwise we would have $\xi \in \partial_\infty B'$. Removing the singular geodesics and $\bigcup_{B' \in B'} \mathscr{B}'$ from H, we get a subset H^0 whose connected components are a countably infinite collection of open strips. If $S \subset H^0$ is such a strip, we let Ω_S be the set of $\xi' \in \Omega$ so that $\overline{p\xi'} \cap S \neq \emptyset$. As in cases I and II, Ω_S is closed and open in Ω. This forces the connected component of ξ in Ω to be contained in $\partial_\infty H \subset \partial_\infty B$, as desired.

7. Vertices and safe paths

We say that $\xi \in \partial_\infty X$ is a *vertex* if there is a neighborhood U of ξ such that the path component of ξ in U is homeomorphic to the cone over a Cantor set, with ξ corresponding to the vertex of the cone. By Lemma 4 the set of vertices in $\bigcup_{B \in \mathscr{B}} \partial_\infty B$ is precisely the set of poles in $\bigcup_{B \in \mathscr{B}} \partial_\infty B$ (a priori there may be other vertices in $\partial_\infty X$).

A path $c:[0,1] \to \partial_\infty X$ is *safe* if $c(t)$ is a vertex for only finitely many $t \in [0,1]$. Since the property of being joinable by a safe path is an equivalence relation on pairs of points, and since $\partial_\infty B_1 \cup \partial_\infty B_2$ is safe path connected when B_1 is adjacent to B_2, it follows that $\bigcup_{B \in \mathscr{B}} \partial_\infty B$ is safe path connected.

Lemma 6. *$\bigcup_{B \in \mathscr{B}} \partial_\infty B$ is a safe path component of $\partial_\infty X$.*

Proof. First note that if $c:[0,1] \to \partial_\infty X$ is a path, $c(t)$ is not a vertex when $t \in (0,1)$, $B \in \mathscr{B}$, and $c(0) \in \partial_\infty B$ is not a pole of any block other than B, then $c([0,1]) \subset \partial_\infty B$. This follows from Lemma 4, the fact that $\partial_\infty B$ is closed in $\partial_\infty X$, and a continuity argument.

Now if $B_0 \in \mathscr{B}$, $c:[0,1] \to \partial_\infty X$ is a safe path starting in $\partial_\infty B_0$, and $0 = t_0 < t_2 \ldots < t_k = 1$ are chosen so that $c(t)$ is a vertex only if $t = t_i$ for some i, then one proves by induction on i that the intervals $[t_{i-1}, t_i]$ are mapped into $\bigcup_{B \in \mathscr{B}} \partial_\infty B$. $\square$

Lemma 7. *Pick $B_0 \in \mathscr{B}$ and $p \in B_0 \setminus \bigcup_{W \in \mathscr{W}} W$. Let c: $[0,1] \to \partial_\infty X$ be a path, and suppose $c(0)$ has an infinite p-itinerary. Then either $c(t)$ has the same p-itinerary as $c(0)$ for all $t \in I$, or there is a $\bar{t} \in I$ so that $c(\bar{t})$ has a finite itinerary. In particular, by Lemma 6, if c is a safe path then $c(t)$ has the same p-itinerary as $c(0)$ for all $t \in I$.*

Proof. Suppose $\xi_k \in \partial_\infty X$ is a sequence with $\lim_{k \to \infty} \xi_k = \xi \in \partial_\infty X$, and a certain block B is in the itinerary of $\overline{p\xi_k}$ for every k. Then either

1. *Itin*(ξ) contains B

   or
2. *Itin*(ξ) is finite and only contains blocks lying between B_0 and B.

To see this, suppose B' is in $Itin(\xi)$ and $x \in \overline{p\xi} \cap Int(B')$. Then $x = \lim_{j \to \infty} x_j$ where $x_j \in \overline{p\xi_j} \cap Int(B')$ for sufficiently large j, so B' is in $Itin(\xi_j)$ for sufficiently large j. This means that B' lies between B_0 and B, for otherwise B would have to lie between B_0 and B', forcing $B \in Itin(\xi)$.

The lemma now follows, since if B is in $Itin(c(0))$ but not in $Itin(c(t))$ for all $t \in [0, 1]$, then setting $t_0 := \inf\{t | B \notin Itin(c(t))\}$ we get a ray $\overline{pc(t_0)}$ with finite itinerary by the reasoning of the preceding paragraph. □

Corollary 8. *There is a unique safe path component of $\partial_\infty X$ which is dense, namely $\bigcup_{B \in \mathscr{B}} \partial_\infty B$.*

Proof. By Lemma 6 we know that $\bigcup_{B \in \mathscr{B}} \partial_\infty B$ forms a safe path component. $\bigcup_{B \in \mathscr{B}} \partial_\infty B$ is dense in $\partial_\infty X$ since any initial segment $\overline{px}$ of a ray $\overline{p\xi}$ may be continued as a ray $\overline{p\xi} = \overline{p} \cup \overline{x\xi'}$ where the continuation $\overline{x\xi'}$ lies in a block (one of at most two) containing x.

By Lemma 7, if $\xi \in \partial_\infty X$ has an infinite p-itinerary, then any safe path starting at ξ consists of points with the same p-itinerary. Clearly the collection of points with a given p-itinerary is not dense in $\partial_\infty X$. The corollary follows. □

8. Detecting block boundaries

Call an arc $I \subset \bigcup_{B \in \mathscr{B}} \partial_\infty B$ an *edge* if its endpoints are both vertices, but no interior point of I is vertex of $\partial_\infty X$. Edges are contained in the boundary of a single block $B \in \mathscr{B}$ (see the proof of Lemma 6). Clearly the endpoints of an edge $I \subset \bigcup_{B \in \mathscr{B}} \partial_\infty B$ are either the poles of a single block, or $I \subset \partial_\infty W$ where $W = B_1 \cap B_2$ and the endpoints of I are poles of B_1 and B_2. So two points in $\bigcup_{B \in \mathscr{B}} \partial_\infty B$ are the poles of a single block (resp. adjacent blocks) iff they are the endpoints of more than one edge (resp. a unique edge). A subset of $\bigcup_{B \in \mathscr{B}} \partial_\infty B$ is the boundary of a block $B \in \mathscr{B}$ iff it is the union of all edges intersecting the poles of B.

9. Limiting behavior of poles

Pick $B \in \mathscr{B}$, and consider the set $\mathscr{P}$ of poles of blocks adjacent to B. If $\eta \in \partial_\infty B$ is a pole of B, then we have $\angle_T(\xi, \eta) \in \{\alpha, \pi - \alpha\}$ for every $\xi \in \mathscr{P}$. Let $\mathscr{P}_\alpha := \{\xi \in \mathscr{P} | \angle_T(\xi, \eta) = \alpha\}$, and $\mathscr{P}_{\pi-\alpha} := \{\xi \in \mathscr{P} | \angle_T(\xi, \eta) = \pi - \alpha\}$. Call each arc of $\partial_\infty B$ joining the poles of B a *longitude*.

Lemma 9. *Each longitude of $\partial_\infty B$ intersects the closure of $\mathscr{P}_\alpha$ (resp. $\mathscr{P}_{\pi-\alpha}$) in a single point ξ with $\angle_T(\xi, \eta) = \alpha$ (resp $\angle_T(\xi, \eta) = \pi - \alpha$).*

Proof. Pick $p \in B$, $\xi \in \partial_\infty B$ with $\angle_T(\xi, \eta) = \alpha$. Any initial segment $\overline{px}$ of the ray $\overline{p\xi}$ may be extended to a segment $\overline{py} = \overline{px} \cup \overline{xy}$ so that $\overline{py} \cap W = \{y\}$ for some wall $W \subset B$. Then $\overline{py}$ may be extended as a ray $\overline{p\xi'} = \overline{py} \cup \overline{y\xi'}$ where $\overline{y\xi'} \subset W$ and $\xi' \in \mathscr{P}_\alpha$. Therefore $\xi \in \bar{\mathscr{P}}_\alpha$. Since $\angle_T(\cdot, \eta)$ is a continuous function on $\partial_\infty B$, each longitude intersects $\mathscr{P}_\alpha$ in a single point. Similar reasoning applies to $\mathscr{P}_{\pi-\alpha}$.

From the lemma we see that any longitude l of $\partial_\infty B$ intersects $\bar{\mathscr{P}}$ in two points if $\alpha < \pi/2$ and one point if $\alpha = \pi/2$.

10. Distinguishing torus complexes

Let $\bar{X}_1$ be a torus complex with $\alpha < \pi/2$, and let $\bar{X}_2$ be a torus complex with $\alpha = \pi/2$. Let X_1 and X_2 be their respective universal covers. A homeomorphism $f: \partial_\infty X_1 \to \partial_\infty X_2$ would carry safe path components to safe path components, block boundaries to block boundaries (Corollary 8 and Section 8), poles to poles, and longitudes to longitudes. But then Section 9 gives a contradiction.

References

[1] S.B. Alexander, R.L. Bishop, The Hadamard-Cartan theorem in locally convex metric spaces, L'Enseignement Mathematica 36 (1990) 309–320.
[2] W. Ballmann, Axial isometries of manifolds of nonpositive curvature, Mathematische Annalen 252 (1982) 131–144.
[3] W. Ballmann, Lectures on spaces of nonpositive curvature, DMV seminar, Band 25, Birkhäuser.
[4] G. Besson, G. Courtois, S. Gallot, Entropies et rigidits des espaces localement symtriques de courbure strictement negative, Geometric and Functional Analysis 5(5) (1995) 731–799.
[5] S. Buyalo, Geodesics in Hadamard spaces, St. Petersburg Math. Journal, to appear.
[6] C. Croke, B. Kleiner, The geodesic flow of a nonpositively curved graph manifold. preprint.
[7] M. Gromov, Hyperbolic groups, in: Essays in Group Theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York-Berlin, 1987 pp. 75–263.
[8] M. Gromov, Asymptotic invariants of infinite groups, in: Niblo and Rollers (Eds), Geometric Group Theory, LMS, 1993.
[9] B. Kleiner, B. Leeb, Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings, Publications IHES, 86 (1997) 115–197.
[10] G.D. Mostow, Strong rigidity of locally symmetric spaces, Annals of Mathematics Studies, vol. 78.
[11] K. Ruane, Ph.D. Thesis, Florida State University, 1996.