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A CAT(0) group with uncountably many distinct boundaries

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Abstract. Croke and Kleiner [5] gave a construction for a family $\{X_\alpha : 0 < \alpha \leqslant \frac{\pi}{2}\}$ of CAT(0) spaces that each admit a geometric action by the same group G. They showed that $\partial X_\alpha \not\approx \partial X_{\pi/2}$ for all $\alpha < \frac{\pi}{2}$. We show that in fact $\partial X_\alpha \not\approx \partial X_\beta$ for all $\alpha \neq \beta$, so that G is a CAT(0) group with uncountably many non-homeomorphic boundaries.

1 Introduction

It is known that the geometric boundary of a hyperbolic group is *unique* in the following strong sense: if a finitely generated group G acts geometrically (i.e. cocompactly, properly discontinuously and by isometries) on two hyperbolic metric spaces X and X', then the quasi-isometry from X to X' induced by these actions extends G-equivariantly to a homeomorphism of their boundaries. In particular, the intrinsically defined boundary of G is homeomorphic to the geometric boundary of any space on which it acts geometrically. For a full treatment of the uniqueness of hyperbolic group boundaries, see [8].

The question of uniqueness of boundaries for the class of CAT(0) groups is still being explored. A group G is said to be CAT(0) if it acts geometrically on some CAT(0) geodesic space X. A CAT(0) group does not have an intrinsically defined boundary, but one might hope to define its boundary to be ∂X. For some CAT(0) groups this is more or less meaningful. For example, if G also happens to be hyperbolic and acts on a CAT(0) space X, then the boundary attached to X as a CAT(0) space agrees with its hyperbolic boundary; this boundary is unique to G in the strong sense described above (see [8, Chapter 7]). As another example, Bowers and Ruane proved that when G' is hyperbolic, then the boundary of any CAT(0) space on which a group of the form $G = G' \times \mathbb{Z}^n$ acts geometrically is homeomorphic to $\partial G' * S^{n-1}$; however in general this homeomorphism cannot be taken to be G-equivariant; see [3]. Finally, it is hoped that some form of boundary uniqueness result can be proved for the class of Coxeter groups, which are CAT(0) by [9]; see also [4], [7].

In 1997 Croke and Kleiner gave an example of a CAT(0) group G with at least two distinct boundaries. They actually constructed an uncountable collection of CAT(0) spaces $\{X_\alpha : 0 < \alpha \leqslant \frac{\pi}{2}\}$ each admitting a geometric action by G, and they showed that $\partial X_{\pi/2} \not\approx \partial X_\alpha$ for all $\alpha < \frac{\pi}{2}$. By generalizing their methods, we show that $\partial X_\alpha \not\approx \partial X_\beta$ for all $\alpha \neq \beta$, so that there are in fact uncountably many different boundaries associated to the group G. This is particularly remarkable because G is a right-angled Artin group which is a finite-index subgroup of a Coxeter group, and which is furthermore a free product with amalgamation of two groups of the form shown by Bowers and Ruane to have unique boundaries.

2 Preliminaries

Before reviewing the Croke and Kleiner construction, we recall the definition of the boundary of a CAT(0) space X. Choose a basepoint $p \in X$ and let $\mathscr{R}(p)$ denote the set of unit-speed geodesic rays issuing from p. To every ray in $\mathscr{R}(p)$, associate an endpoint $\sigma(\infty)$. For $r \geqslant 0$ and $\varepsilon > 0$, two rays $\sigma, \tau \in \mathscr{R}(p)$ are (r, ε)-*close* if $d(\sigma(r), \tau(r)) < \varepsilon$. This defines a topology (the *cone topology*) on the set $\{\sigma(\infty) : \sigma \in \mathscr{R}(p)\}$. This space is the boundary ∂X of X, and up to homeomorphism it is independent of the choice of basepoint p. See [2].

For any two geodesic segments or rays σ and τ with a common point x in a CAT(0) space X, we denote by $\angle_x(\sigma, \tau)$ the Alexandrov angle at x between σ and τ; see [2, Section II.3]. More generally, if σ and τ are rays in $\mathscr{R}(p)$, let $\angle_x(\sigma, \tau) = \angle_x(\sigma', \tau')$, where $\sigma', \tau' \in \mathscr{R}(x)$ and σ', τ' are asymptotic with σ and τ respectively.

The space X_α constructed by Croke and Kleiner is the universal cover of a *torus complex* $\overline{X}_\alpha$, constructed as follows. Start with a flat torus T_0 with the property that a pair a_1, a_2 of closed, π_1-generating geodesics in T_0 meet in a single point at an angle α, with $0 < \alpha \leqslant \frac{\pi}{2}$. Note that any angle $\alpha \in \left(0, \frac{\pi}{2}\right]$ can be realized by taking two lines in $\mathbb{R}^2$ that meet at the angle α and identifying parallel copies of these lines in the modeling of T_0. For $i = 1, 2$, let T_i be a flat torus containing an essential loop b_i, such that $\mathrm{length}(b_i) = \mathrm{length}(a_i)$, and let $\overline{X}_\alpha$ be the union of T_0, T_1, and T_2, where a_i is identified with b_i. For $i = 1, 2$, let $\overline{Y_i} = T_0 \cup T_i \subset \overline{X}_\alpha$.

The following facts about X_α (the universal cover of $\overline{X}_\alpha$) were established in [5].

1 Blocks. A connected component of the preimage of $\overline{Y_i}$ in X_α is called a *block*. Each block is a copy of the universal cover of $\overline{Y_i}$, and hence is isometric to the metric product of a simplicial valence-4 tree with $\mathbb{R}$. At each vertex of the tree, the $\mathbb{R}$ factor is a lift of a_i.

2 Walls. Each block is a tree of planes of two types, and a plane of the type that covers T_0 is referred to as a *wall* of the block. Each wall is common to exactly two blocks (one covering $\overline{Y_1}$ and the other covering $\overline{Y_2}$), which are called *adjacent* blocks. Any two blocks of X_α either are disjoint or are adjacent with a wall as their only intersection. Blocks and walls are convex subsets of X_α.

The *nerve* $\mathcal{N}(X_\alpha)$ of X_α is the graph that has one vertex for every block of X_α and which satisfies the property that vertices are adjacent exactly when the corresponding blocks are adjacent. The nerve of X_α is in fact a tree.

3 Block boundaries. Given a block B, its boundary ∂B embeds in ∂X_α and is homeomorphic to the suspension of a Cantor set. Let B cover $\overline{Y}_i$; note that the lifts of a_i in B are parallel geodesic lines. The two suspension points of ∂B, called *poles* of the block, are the common endpoints of these lifts. A *longitude* of the block is an arc in ∂B joining the two poles, i.e. the suspension of a point in the Cantor set. If two blocks are adjacent with common wall W, then their boundaries meet exactly in ∂W. If they are at distance 2 in the nerve, then their boundaries meet exactly in the two poles of the block between them. If they are at distance 3 or more in the nerve, then their boundaries are disjoint.

We say that a geodesic ray σ *enters a plane* V if there are values $r < R$ in the domain of σ such that $\sigma([r, R]) \subset V$, and that σ *enters a block* B if it enters a non-wall plane of B.

It should be noted that the set of block boundaries does not exhaust ∂X_α, which also includes the endpoints of rays which pass through infinitely many blocks. However, since the set of block boundaries is invariant (see next item) and has enough structure to distinguish between values of α, we will not investigate beyond this subset.

4 Topological invariance. Note that the metric on X_α depends on the choice of α. Thus for $\alpha, \beta \in \left(0, \frac{\pi}{2}\right]$, X_α and X_β are homeomorphic but not isometric, hence not equivalent as CAT(0) spaces, and their boundaries are not necessarily homeomorphic. Any homeomorphism from ∂X_α to ∂X_β must take block boundaries to block boundaries, poles to poles, and longitudes to longitudes.

Note that a homeomorphism of the boundary preserves poles, block boundaries, and block adjacency. Thus for any block B, the closure $\mathscr{P}$ of the set of poles of adjacent blocks is preserved under a homeomorphism. Croke and Kleiner showed that any longitude of ∂B meets $\mathscr{P}$ in one point if $\alpha = \frac{\pi}{2}$ and two points if $\alpha < \frac{\pi}{2}$, thus proving that $\partial X_\alpha \not\approx \partial X_{\pi/2}$ for all $\alpha < \frac{\pi}{2}$. We will extend this strategy inductively to prove that $\partial X_\alpha \not\approx \partial X_\beta$ whenever $\alpha \neq \beta$.

To do this, we will need to be able to compute exactly where a longitude meets $\mathscr{P}$ and related sets. To this end, we will exploit the natural metric on each block boundary (that it acquires as a spherical join) to identify the *latitudes* of a block boundary, which are level sets ranging from the north pole to the south pole. When $\alpha < \frac{\pi}{2}$, the set $\mathscr{P}$ consists of two latitudes, and these decompose ∂B into three subsets which we can think of as an equatorial band and two polar caps. When $\alpha = \frac{\pi}{2}$, these latitudes coincide as the equator. We shall go on to define related sets $\mathscr{S}_k$ for every positive integer k, and we shall see that these sets are also comprised of pairs of latitudes in ∂B. When α is a rational multiple of π, any longitude will meet the union of these special latitudes in a finite number of points; the number and configuration of these points are topological invariants that determine α. When α is an irrational multiple of π, a longitude will meet this union in a dense countable subset; now the configuration of these points can be used to define a non-standard ordering of the positive integers, which again is a topological invariant that determines α.

Let $G = \pi_1(\overline{X}_\alpha)$ be the fundamental group of the base space. Since these torus complexes all belong to the same homeomorphism class, G does not depend on α. The natural action of G on X_α by deck transformations is a geometric action, and so G is a CAT(0) group. Each of the boundaries ∂X_α is said to be a boundary of G. The results of this paper imply that G has uncountably many distinct boundaries.

We begin by examining the metric on a block boundary. Let B be a block and V a plane of B. Then $d_{\partial V}(\sigma(\infty), \tau(\infty)) = \angle_x(\sigma, \tau)$ defines a metric on ∂V compatible with the cone topology, where x is any point in V. If Q_0 and Q_π denote the poles of B, then $d_{\partial V}(Q_0, Q_\pi) = \pi$. Recall that ∂B is the suspension of a Cantor set, i.e. the spherical join of a two-point set (Q_0 and Q_π) and a Cantor set. If we equip the Cantor set with a metric and take the distance between Q_0 and Q_π to be π, then this yields a natural metric $d_{\partial B}$ on ∂B in which each longitude is parametrized by $[0, \pi]$. (See [2, (1.5.13), p. 63].) Each plane boundary is isometrically embedded in ∂B with this metric. Thus if W is a wall of B common to B and B', then the poles of B' are at distance α from Q_0 and Q_π respectively. As a ray σ travels from plane to plane of a block B, its angle of incidence with the geodesics joining the poles is preserved. Let $t \in (r, R]$, where $\sigma([r, R]) \subset B$. Then the points in the shadow of $\sigma(t)$ on ∂B (the endpoints in ∂B of rays which coincide with σ at least as far as t) all lie at the same distance from Q_0.

Note that as a ray travels from one *block* to another, its angle of incidence with the poles is *not* necessarily preserved, even up to a difference of α. In fact, if the ray travels from a non-wall V of one block directly into a non-wall V' of the next via a point $\sigma(t)$, the shadow of $\sigma(t)$ on $\partial V'$ is a pair of arcs rather than a pair of points. See the Transformation Rules in [1] for a more detailed analysis of this phenomenon.

Definition 2.1. Given a block B and $\theta \in [0, \pi]$, the *θ-latitude of ∂B* is the set

$$\mathscr{L}_\theta(\partial B) = \{Q \in \partial B : d_{\partial B}(Q, Q_0) = \theta\}.$$

Note that each longitude of ∂B meets each latitude in exactly one point, and ∂B is the disjoint union of the latitudes $\{\mathscr{L}_\theta(\partial B) : 0 \leqslant \theta \leqslant \pi\}$.

More generally, for all $\phi \in \mathbb{R}$, let $\mathscr{L}_\phi(\partial \mathscr{B}) = \mathscr{L}_\theta(\partial \mathscr{B})$ where $\theta \in [0, \pi]$ and $\phi \equiv \pm\theta \pmod{2\pi}$.

Let W be a wall of block B. It will be convenient for our induction argument in Proposition 2.4 to label points of ∂W with real-valued indices indicating their location within ∂W in a way that is compatible from block to block. First, let Q_α and $Q_{\pi+\alpha}$ denote the two points in ∂W which are poles of an adjacent block, at distance α from Q_0 and Q_π respectively. Let L_+ and L_- be the two longitudes which comprise ∂W, where $Q_\alpha \in L_+$ and $Q_{\pi+\alpha} \in L_-$. For all $\theta \in (-\pi, \pi]$, define

$$Q_\theta = \begin{cases} L_- \cap \mathscr{L}_\theta(\partial \mathscr{B}) & \text{if } -\pi < \theta \leqslant 0, \\ L_+ \cap \mathscr{L}_\theta(\partial \mathscr{B}) & \text{if } 0 < \theta \leqslant \pi. \end{cases}$$

In general, let $Q_{\theta+2k\pi} = Q_\theta$ for all $k \in \mathbb{Z}$. Then $Q_\phi \in \mathscr{L}_\theta(\partial \mathscr{B})$ if and only if

$\phi \equiv \pm\theta \pmod{2\pi}$. Suppose that $W = B \cap B'$. The index of Q as it lies in ∂B differs by α from its index as it lies in $\partial B'$. In fact, by choosing the labeling of the poles Q_0 and Q_π of each block carefully, we can ensure that $Q_\theta(W, B) = Q_{\alpha-\theta}(W, B')$ for all $\theta \in \mathbb{R}$. See Figure 1.

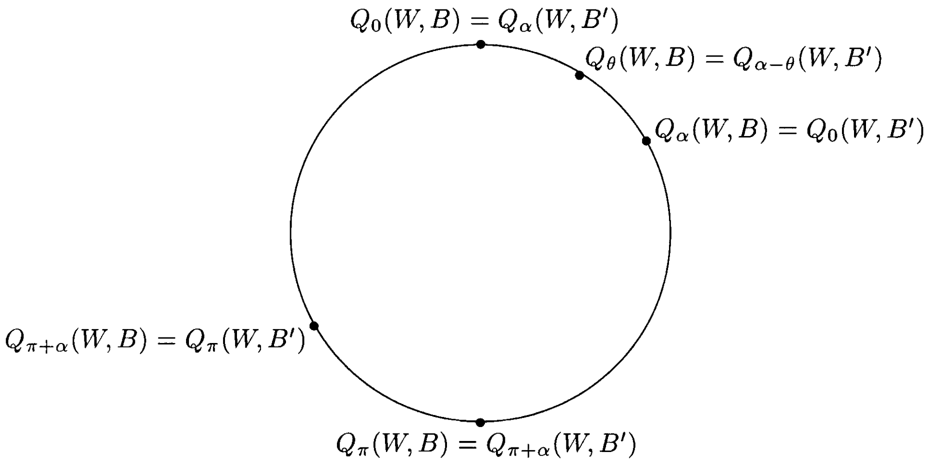

Figure 1. $Q_\theta(W, B) = Q_{\alpha-\theta}(W, B')$

Proposition 2.2. *Let B be a block and $\theta \in [0, \pi]$. The set of all points at distance θ from Q_0, which lie in the boundaries of walls of B, is dense in $\mathcal{L}_\theta(\partial B)$.*

Proof. Given $Q = \sigma(\infty) \in \mathcal{L}_\theta(\partial\mathcal{B})$ and $r > 0$, we find a ray $\tau \in \mathcal{R}(p)$ such that σ and τ are (r, ε)-close for all $\varepsilon > 0$, $\tau(\infty) \in \mathcal{L}_\theta(\partial\mathcal{B})$, and $\tau(\infty)$ lies in a wall boundary. Choose $R_2 > R_1 \geqslant r$ such that $\sigma([R_1, R_2])$ lies in a plane V of B; we require that τ coincide with σ at least as far as R_2. If V is a wall, we allow τ to remain in V thereafter. If V is a non-wall, then τ enters a wall W of B at some point $\tau(R_3) = \sigma(R_3)$ with $R_3 \geqslant R_2$, and thereafter travels only in W. In each case, $\tau(\infty)$ lies in the shadow of $\sigma(R_2)$ on ∂B, hence in $\mathcal{L}_\theta(\partial\mathcal{B})$.

Let $\mathcal{P}(\partial B)$ denote the closure of the set of poles of blocks adjacent to B.

Proposition 2.3. *$\mathcal{P}(\partial B) = \mathcal{L}_\alpha(\partial B) \cup \mathcal{L}_{\pi-\alpha}(\partial B)$ for any block B. Thus if $h : \partial X_\alpha \to \partial X_\beta$ is a homeomorphism, then $h(\mathcal{L}_\alpha(\partial B) \cup \mathcal{L}_{\pi-\alpha}(\partial B)) = \mathcal{L}_\beta(h(\partial B)) \cup \mathcal{L}_{\pi-\beta}(h(\partial B))$.*

Proof. If P is a pole of an adjacent block, then $P = Q_\alpha(W, B)$ or $P = Q_{\pi+\alpha}(W, B)$ for some wall W of B. Thus the closure of the set of such points is exactly $\mathcal{L}_\alpha(\partial B) \cup \mathcal{L}_{\pi+\alpha}(\partial B) = \mathcal{L}_\alpha(\partial B) \cup \mathcal{L}_{\pi-\alpha}(\partial B)$. The topological invariance follows from that of block boundaries, poles, and block adjacency.

Note that the latitudes $\mathscr{L}_\alpha(\partial B)$ and $\mathscr{L}_{\pi-\alpha}(\partial B)$ coincide if and only if $\alpha = \frac{\pi}{2}$. This is a restatement, using the vocabulary of latitudes, of the fact from which Croke and Kleiner concluded that $\partial X_\alpha \not\approx \partial X_{\pi/2}$ for all $\alpha < \frac{\pi}{2}$.

Proposition 2.4. *Let $h : \partial X_\alpha \to \partial X_\beta$ be a homeomorphism, $B \subset X_\alpha$ a block, and $W \subset B$ a wall. Let D be a block of X_β and $V \subset D$ a wall such that $h(\partial B) = \partial D$ and $h(\partial W) = \partial V$. Then the following assertions hold for all integers $k \geqslant 0$:*

(1) *if $\theta \in \{k\alpha, \pi + k\alpha\}$, then $h(Q_\theta(W,B)) = Q_\phi(V,D)$ where $\phi \in \{k\beta, \pi + k\beta\}$;*

(2) *if $\theta \in \{-k\alpha, \pi - k\alpha\}$, then $h(Q_\theta(W,B)) = Q_\phi(V,D)$ where $\phi \in \{-k\beta, \pi - k\beta\}$.*

Proof. The case $k = 0$ is trivial. For $k \geqslant 1$, we will proceed by induction on k.

The case $k = 1$ follows from the proof of Proposition 2.3 and the fact that of the four points in question, two are poles (of adjacent blocks) and two are not.

Assume that $k > 1$, and let $W = B \cap B'$. Let D' be a block of X_β such that $h(\partial B') = \partial D'$; then $V = D \cap D'$, where $h(\partial W) = \partial V$. Recall that for all $\theta \in \mathbb{R}$ we have $Q_\theta(W,B) = Q_{\alpha-\theta}(W,B')$. By induction,

$$\begin{aligned} h(Q_{k\alpha}(W,B)) &= h(Q_{\alpha-k\alpha}(W,B')) \\ &= h(Q_{-(k-1)\alpha}(W,B')) \in \{Q_{-(k-1)\beta}(V,D'), Q_{\pi-(k-1)\beta}(V,D')\} \\ &\qquad = \{Q_{k\beta}(V,D), Q_{\pi+k\beta}(V,D)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} h(Q_{\pi+k\alpha}(W,B)) = h(Q_{\pi-(k-1)\alpha}(W,B')) &\in \{Q_{-(k-1)\beta}(V,D'), Q_{\pi-(k-1)\beta}(V,D')\} \\ &= \{Q_{k\beta}(V,D), Q_{\pi+k\beta}(V,D)\}. \end{aligned}$$

We can now assume that the first statement of the proposition holds in all walls of B. Passing to the closure of these sets and referring to Proposition 2.2 we conclude that

$$h(\mathscr{L}_{k\alpha}(\partial B) \cup \mathscr{L}_{\pi-k\alpha}(\partial B)) \subset \mathscr{L}_{k\beta}(\partial D) \cup \mathscr{L}_{\pi-k\beta}(\partial D). \tag{1}$$

The second statement of the proposition now follows because h takes ∂W homeomorphically to ∂V.

Let B be a block of X_α. For every integer $k \geqslant 0$, let $\mathscr{S}_k(\partial B)$ denote the set $\mathscr{L}_{k\alpha}(\partial B) \cup \mathscr{L}_{\pi-k\alpha}(\partial B)$, and $\mathscr{S}(\partial B) = \bigcup_{k\geqslant 0} \mathscr{S}_k$. Then Equation (1) above implies the following result.

Corollary 2.5. *For every block B, the sets $\mathscr{S}_k(\partial B)$ are each preserved by a homeomorphism of the boundary, for all $k \geqslant 0$. Thus $\mathscr{S}(\partial B)$ is also invariant.*

3 The rational case

When α is a rational multiple of π, each longitude of a block boundary meets $\mathscr{S}(\partial B)$ in finitely many points.

Proposition 3.1. *If $\alpha = p\pi/q$, where $(p, q) = 1$, then any longitude of ∂B meets the set $\mathscr{S}(\partial B)$ in $q + 1$ distinct points.*

Proof. Throughout, let $\equiv$ denote congruence modulo 2π. First note that for all $k \in \mathbb{Z}$ we have $k(\pi/q) \equiv \pm s(\pi/q)$ for some $0 \leqslant s \leqslant q$: Letting $k = qd + r$, where $d, r \in \mathbb{Z}$ and $0 \leqslant r < q$, we see that $k(\pi/q) = d\pi + r(\pi/q)$, which is congruent to $r(\pi/q)$ or $\pi + r(\pi/q)$ depending on whether d is even or odd. Furthermore, $\pi + r(\pi/q) \equiv \pm(\pi - r(\pi/q)) \equiv \pm(q - r)(\pi/q)$, where $0 < q - r \leqslant q$. Next note that since $\alpha = p(\pi/q)$, then for all $k \in \mathbb{Z}$ we have $k\alpha \equiv \pm s\alpha$ for some s with $0 \leqslant s \leqslant q$. Thus to prove the proposition, it suffices to show that

$$\Bigl(\bigcup_{0 \leqslant k \leqslant q} \mathscr{L}_{k\alpha}(\partial B)\Bigr) \cup \Bigl(\bigcup_{0 \leqslant k \leqslant q} \mathscr{L}_{\pi - k\alpha}(\partial B)\Bigr) = \bigcup_{0 \leqslant k \leqslant q} \mathscr{L}_{k(\pi/q)}(\partial B),$$

as the former set would account for all of $\mathscr{S}(\partial B)$ and the latter set clearly comprises $q + 1$ distinct latitudes in ∂B. Recall that if $\theta \equiv \pm\phi$, then $\mathscr{L}_\theta(\partial B) = \mathscr{L}_\phi(\partial B)$.

$\subseteq$: For all k we can write $k\alpha = kp(\pi/q) \equiv \pm s(\pi/q)$ for some s with $0 \leqslant s \leqslant q$, and

$$\pi - k\alpha = \left(1 - \frac{kp}{q}\right)\pi = (q - kp)\left(\frac{\pi}{q}\right) \equiv \pm s\left(\frac{\pi}{q}\right)$$

for some $0 \leqslant s \leqslant q$.

$\supseteq$: Since $(p, q) = 1$, there are integers m and n such that $mq + np = 1$, and hence $m\pi + n\alpha = \pi/q$. Thus

$$k\left(\frac{\pi}{q}\right) = km\pi + kn\alpha \equiv \pm s\alpha \text{ or } \pm(\pi - s\alpha)$$

for some s with $0 \leqslant s \leqslant q$, depending on whether km is even or odd.

Note that longitudes of ∂B which lie in wall boundaries are topologically distinguished by the fact that they alone contain exactly three poles: the endpoints (which are poles of ∂B) and a point in the interior (which is a pole of an adjacent block). The pole in the interior will lie in $\mathscr{L}_\alpha(\partial B)$ or $\mathscr{L}_{\pi-\alpha}(\partial B)$, and hence is separated from the endpoints by $p - 1$ non-pole elements of $\mathscr{S}(\partial B)$ on one side and $(q - 2) - (p - 1)$ non-pole elements of $\mathscr{S}(\partial B)$ on the other. See Figure 2. This arrangement is preserved by a homeomorphism. In this way we see that both p and q are topological invariants of ∂X_α when $\alpha = p\pi/q$ is a rational multiple of π, and this proves the theorem in this case:

Theorem 3.2. *If α and β are both rational multiples of π and $\partial X_\alpha \approx \partial X_\beta$, then $\alpha = \beta$.*

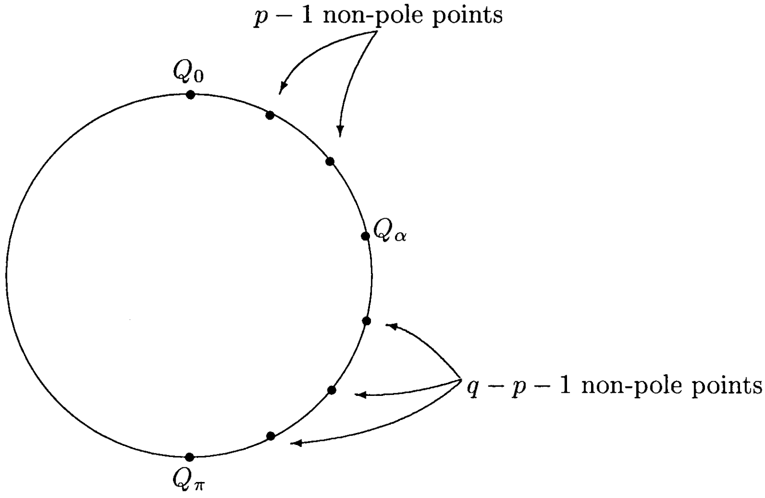

Figure 2. $\alpha = \frac{3}{7}\pi$, $p = 3$, $q = 7$

4 The irrational case

When α is an irrational multiple of π, then the sets $\mathscr{S}_k$, $k \geqslant 0$, are dense in a block boundary. In particular, each longitude meets $\mathscr{S}(\partial B)$ in a dense countable subset. We can immediately conclude that ∂X_α and ∂X_β cannot be homeomorphic if α is a rational multiple of π and β an irrational multiple. The method used before of counting points of $\mathscr{S}(\partial B)$ in a longitude will no longer suffice to distinguish between different irrational multiples of π, and we must examine a more subtle property of their configuration.

Note that for every longitude L of ∂B and every $k \geqslant 0$, the set

$$L \cap \mathscr{S}_k(\partial B) = L \cap (\mathscr{L}_{k\alpha}(\partial B) \cup \mathscr{L}_{\pi - k\alpha}(\partial B))$$

contains exactly two points, one in the upper half of L and the other in the lower half. Thus our choice of α induces a non-standard ordering of the non-negative integers $\mathbb{Z}^+$ as follows. Let $f : \mathbb{Z}^+ \to L$ be the function taking k to the unique point in the upper half of L (i.e. at distance at most $\frac{\pi}{2}$ from Q_0) in $\mathscr{S}_k(\partial B)$. For all integers $k, l \geqslant 0$, write $k \prec_\alpha l$ if and only if $d_{\partial B}(Q_0, f(k)) < d_{\partial B}(Q_0, f(l))$. Note that $0 \prec_\alpha k$ for all $k > 0$. The linear order $\prec_\alpha$ does not depend on our choice of block B or longitude L. Alternatively, we could have used the lower half of L in defining the induced ordering. This would have yielded the exact reversal of $\prec_\alpha$. Thus we can consider $\prec_\alpha$ as being well defined up to reversal.

Proposition 4.1. *Let α and β be irrational multiples of π. Then any homeomorphism $h : \partial X_\alpha \to \partial X_\beta$ preserves the induced ordering of $\mathbb{Z}^+$ up to reversal. That is, $\prec_\beta$ either agrees with $\prec_\alpha$ or is its exact reversal.*

Proof. Let B be a block of ∂X_α, L a longitude of ∂B, and let L^+ and L^- denote its upper half and lower half respectively. Then $h(L) = L'$ is a longitude of $h(\partial B)$. Since h preserves the sets $\mathscr{S}_k(\partial B)$, then L^+ is taken homeomorphically to an arc in L' containing an endpoint of L' and exactly one point in $L' \cap (\mathscr{L}_{k\beta}(h(\partial B)) \cup \mathscr{L}_{\pi - k\beta}(h(\partial B)))$ for all $k \geqslant 0$, i.e. either L'^+ or L'^-. In the first case the ordering is preserved; in the second it is exactly reversed.

Proposition 4.2. *Let α and β be irrational multiples of π. If $\prec_\alpha$ and $\prec_\beta$ are the same up to reversal, then $\alpha = \beta$.*

To prove this, we need a lemma:

Lemma 4.3. *If a, b are irrational numbers such that $0 < a < b < \frac{1}{2}$, then there are integers $j > 2$ and n such that*

$$\frac{n-1}{j-2} < a < \frac{n}{j} < b < \frac{n}{j-2}.$$

Proof. Let $j \geqslant 1$ be the least integer such that $[a, b]$ contains an element of $\{n/j : n \text{ an integer}\}$. Since a and b are irrational we have $a < n/j < b$. Since $0 < n/j < \frac{1}{2}$ we also have $j > 2n \geqslant 2$. Note that $(n-1)/(j-2)$ and $n/(j-2)$ do not lie in $[a, b]$ by the minimality of j. Since $n/j < n/(j-2)$ we have $b < n/(j-2)$. Similarly, $a > (n-1)/(j-2)$, since

$$j > 2n \Rightarrow jn - j < jn - 2n \Rightarrow j(n-1) < n(j-2) \Rightarrow \frac{n}{j} > \frac{n-1}{j-2}.$$

Proof of Proposition 4.2. Suppose instead that $\alpha < \beta$, and let $a = \alpha/\pi$, $b = \beta/\pi$. Then $0 < a < b < \frac{1}{2}$ and so there are integers $j > 2$ and n such that

$$\frac{n-1}{j-2} < a < \frac{n}{j} < b < \frac{n}{j-2}.$$

Setting $k = j - 1$ we have $k > 1$ and

$$\frac{n-1}{k-1} < a < \frac{n}{k+1} < b < \frac{n}{k-1}.$$

Thus

$$\frac{(n-1)\pi}{k-1} < \alpha < \frac{n\pi}{k+1} < \beta < \frac{n\pi}{k-1}. \tag{2}$$

The first and second inequalities in Equation (2) imply that

$$(n-1)\pi + \alpha < k\alpha < n\pi - \alpha. \tag{3}$$

The third and fourth inequalities imply that

$$n\pi - \beta < k\beta < n\pi + \beta. \tag{4}$$

Pictorially, a point with label $k\alpha$ lies in the equatorial band, and a point with label $k\beta$ lies in a polar cap. Thus Equation (3) yields that $0 \prec_\alpha 1 \prec_\alpha k$, whereas Equation (4) yields that $0 \prec_\beta k \prec_\beta 1$, so that these orderings are not equivalent up to reversal, a contradiction.

When α is an irrational multiple of π, then it induces an ordering of $\mathbb{Z}^+$ that is a topological property of ∂X_α and that is unique to α. This proves the theorem in this case:

Theorem 4.4. *Let α and β be irrational multiples of π. If $\partial X_\alpha \approx \partial X_\beta$, then $\alpha = \beta$.*

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