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Coxeter groups are virtually special ☆

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Abstract

In this paper we prove that every finitely generated Coxeter group has a finite index subgroup that is the fundamental group of a special cube complex. Some consequences include: Every f.g. Coxeter group is virtually a subgroup of a right-angled Coxeter group. Every word-hyperbolic Coxeter group has separable quasiconvex subgroups.

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1. Introduction

Since their introduction in [8] as a source of examples, CAT(0) cube complexes have emerged as an increasingly central class of spaces in geometric group theory. In particular, interesting results were obtained recently using Sageev's thesis [21] to cubulate various groups by finding codimension-1 subgroups. We use the term "cubulate" to mean the production of a proper group action on a CAT(0) cube complex.

Coxeter groups were cubulated in [18]. Certain small-cancellation groups were cubulated in [24]. Word-hyperbolic graphs of free groups with cyclic edge groups were cubulated in [14]. *Wallspaces* were introduced in [10] and include CAT(0) cube complexes as main examples. In

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response to the cubulation theorems discussed above, (groups acting on) wallspaces were cubulated in [4,19].

In a previous paper [11], we developed a theory of "special cube complexes" which are nonpositively curved cube complexes whose hyperplanes embed and avoid certain illegal configurations. We showed in [11] that special cube complexes are intimately related to right-angled Coxeter groups and right-angled Artin groups, and in particular if X is a nonpositively curved special cube complex, then $\pi_1 X$ is a subgroup of a right-angled Artin group, and thus a subgroup of a right-angled Coxeter group (see [15,6]).

On any group G the collection of all cosets of finite index subgroups is the basis for a topology, called the *profinite topology*. The operations of multiplication and inversion are continuous with respect to this topology. A subset $H \subset G$ is *separable* if it is closed in the profinite topology of G. In particular, a subgroup $H \subset G$ is separable if and only if H is the intersection of finite index subgroups.

Among the results we obtained in [11] is the following:

Proposition 1.1. *Let X be a compact nonpositively curved cube complex. Then*

(1) *X has a special finite cover if and only if each double hyperplane coset in $\pi_1 X$ is separable.*
(2) *If $\pi_1 X$ is word-hyperbolic then X has a special finite cover if and only if each quasiconvex subgroup of $\pi_1 X$ is separable.*

Proposition 1.1 reveals special cube complexes as a potential nexus between geometric group theory and linearity through subgroup separability. Given a group G, one first finds a system of codimension-1 subgroups (a significant task) to obtain a proper and cocompact action of G on a CAT(0) cube complex $\widetilde{X}$ with $X = G\backslash\widetilde{X}$. One then attempts to prove the separability of the hyperplane double cosets (or general quasiconvex subgroups). Because of Theorem 1.1, separability yields a finite special cover $\hat{X}$. Thus $\pi_1\hat{X}$ is a subgroup of a right-angled Artin group, and hence G is a subgroup of $SL_n(\mathbb{Z})$ for some n.

This intriguing proof scheme has already been carried out successfully in [14], where word-hyperbolic graphs of free groups are shown to be subgroups of $SL_n(\mathbb{Z})$, thus solving a long-standing problem of G. Baumslag concerning their linearity. The treatment there applies to a variety of groups including limit groups.

The object of this paper is to implement this scheme for all finitely generated Coxeter groups. With this goal, we examine the Niblo–Reeves cubulation of the Coxeter group G mentioned above. However, while the Niblo–Reeves action on a CAT(0) cube complex is cocompact when G is word-hyperbolic, it is not always cocompact in general, though it is *cofinite* in the sense that there are finitely many G-orbits of hyperplanes.

On the level of Coxeter groups, the double coset separability that we require corresponds to the separability of sets $H_a H_b$ where H_i is the stabilizer of some geometric wall W_i in the Coxeter complex, and W_a and W_b are walls that cross. A somewhat more general condition than the separability of $H_a H_b$ turns out to be more naturally achieved, and yet equivalent to the original one.

Our main theorem is then:

Theorem 1.2. *Let G be a f.g. Coxeter group, and let C be the Niblo–Reeves* CAT(0) *cube complex upon which G acts properly.*

Then G has a finite index torsion-free subgroup F such that $F\backslash C$ is special.

A *Coxeter group* is given by a presentation $\langle s_i \mid (s_i s_j)^{m_{ij}} = 1\rangle$, where $m_{ii} = 1$ for each i and $2 \leqslant m_{ij} = m_{ji} \leqslant \infty$ for $i \neq j$, where the convention $m_{ij} = \infty$ indicates there is no relation between s_i and s_j. The Coxeter group is *right-angled* if each $m_{ij} \in \{1, 2, \infty\}$.

Our first application is the following result for which we do not know of a direct proof:

Corollary 1.3. *Every f.g. Coxeter group is virtually a subgroup of a f.g. right-angled Artin group, and hence virtually a subgroup of a right-angled Coxeter group.*

Our second application continues a line of work initiated by Scott in [22] (see also [1,9]) who proved that surface groups are subgroup separable, by showing that right-angled reflection groups of the hyperbolic plane have separable quasiconvex subgroups. The details of the proof have allowed little if any progress without the right-angled hypothesis.

Corollary 1.4. *Let G be a word-hyperbolic Coxeter group. Then every quasiconvex subgroup of G is separable.*

The two corollaries follow by combining Theorem 1.2 with Proposition 3.2.

2. CAT(0) cube complexes

2.1. Definitions

An *n-cube* is a copy of $[-1, 1]^n$, and a 0*-cube* is a single point. We regard the boundary of an n-cube as consisting of the union of lower-dimensional cubes. A *cube complex* is a cell complex formed from cubes, such that the attaching map of each cube is combinatorial in the sense that it sends cubes homeomorphically to cubes by a map modelled on a combinatorial isometry of n-cubes. The *link* of a 0-cube v is the complex whose 0-simplices correspond to ends of 1-cubes adjacent to v, and these 0-simplices are joined up by n-simplices for each corner of $(n+1)$-cube adjacent to v.

A *flag complex* is a simplicial complex with the property that any finite pairwise adjacent collection of vertices spans a simplex. A cube complex C is *nonpositively curved* if $\text{link}(v)$ is a flag complex for each 0-cube $v \in C^0$. Simply-connected nonpositively curved cube complexes are called CAT(0) *cube complexes*, and in fact, they admit a CAT(0) metric where each cube is isometric to $[-1, 1]^n \subset \mathbb{R}^n$, however we shall rarely use this metric.

2.2. Right-angled Artin groups

Let Γ be a simplicial graph. The *right-angled Artin group* or *graph group* $G(\Gamma)$ associated to Γ is presented by:

$$\bigl\langle v\colon\ v \in \text{vertices}(\Gamma) \bigm| [u, v]\colon\ (u, v) \in \text{edges}(\Gamma)\bigr\rangle.$$

For our purposes, the most important example of a nonpositively curved cube complex arises from a right-angled Artin group. This is the cube complex $C(\Gamma)$ containing a torus T^n for each copy of the complete graph $K(n)$ appearing in Γ. Note that the torus T^n is isomorphic to the usual product $(S^1)^n$ obtained by identifying opposite faces of an n-cube. We note that $\pi_1 C(\Gamma) \cong G(\Gamma)$ since the 2-skeleton of $C(\Gamma)$ is the standard 2-complex of the presentation above (see for example [3]).

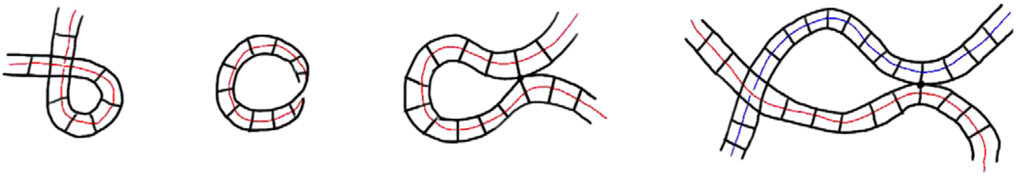

Fig. 1. Immersed hyperplane pathologies.

3. Special cube complexes

3.1. Hyperplanes

A *midcube* of the n-cube $[-1, 1]^n$ is the subspace obtained by restricting exactly one of the coordinates to 0. A *hyperplane* Y in the CAT(0) cube complex C, is a connected subspace whose intersection with each cube is either a midcube or is empty. The 1-cubes intersected by Y are *dual* to Y. For a CAT(0) cube complex, there exists a hyperplane dual to each 1-cube, and moreover, hyperplanes are themselves CAT(0) cube complexes with respect to the cell structure induced by intersection, and are convex subspaces in the CAT(0) metric [21].

We now define an immersed hyperplane in an arbitrary cube complex C. Let M denote the disjoint union of the collection of midcubes of cubes of C. Let D denote the quotient space of M induced by identifying faces of midcubes under the inclusion map. The connected components of D are the *immersed hyperplanes* of C.

3.2. Hyperplane definition of special cube complex

We shall define a special cube complex as a nonpositively curved cube complex which does not have certain pathologies related to its immersed hyperplanes. (See Fig. 1.)

An immersed hyperplane D *crosses itself* if it contains two different midcubes from the same cube of C.

An immersed hyperplane D is *2-sided* if the map $D \to C$ extends to a map $D \times I \to C$ which is a combinatorial map of cube complexes.

A 1-cube of C is *dual* to D if its midcube is a 0-cube of D. When D is 2-sided, it is possible to consistently orient its dual 1-cubes so that any two dual 1-cubes lying (opposite each other) in the same 2-cube are oriented in the same direction.

An immersed 2-sided hyperplane D *self-osculates* if for one of the two choices of induced orientations on its dual 1-cells, some 0-cube v of C is the initial 0-cube of two distinct dual 1-cells of D.

A pair of distinct immersed hyperplanes D, E *cross* if they contain distinct midcubes of the same cube of C. We say D, E osculate, if they have dual 1-cubes which contain a common 0-cube, but do not lie in a common 2-cube. Finally, a pair of distinct immersed hyperplanes D, E *inter-osculate* if they both cross and *osculate*, meaning that they have dual 1-cubes which share a 0-cube but do not lie in a common 2-cube.

A cube complex is *special* if all the following hold:

(1) No immersed hyperplane crosses itself;
(2) Each immersed hyperplane is 2-sided;

(3) No immersed hyperplane self-osculates;
(4) No two immersed hyperplanes inter-osculate.

Example 3.1. Any graph is special. Any CAT(0) cube complex is special. The cube complex associated to a right-angled Artin group is special (see [11]).

3.3. Right-angled Artin group characterization

We give the following characterization of special cube complexes in [11]:

Proposition 3.2. *A cube complex is special if and only if it admits a combinatorial local isometry to the cube complex of a right-angled Artin group.*

A quick explanation of Proposition 3.2 is that for a local isometry $B \to C$, the prohibited hyperplane pathologies on B map to the same prohibited pathologies in C. On the other hand, if C is special, then we define a graph Γ whose vertices are the immersed hyperplanes of C, and whose edges correspond to intersecting hyperplanes. Then there is a natural map $C \to C(\Gamma)$ which is a local isometry.

3.4. Properties

As shown in [11], fundamental groups of special cube complexes have some interesting properties, which we record as follows:

Proposition 3.3. *Let X be a special cube complex with finitely many immersed hyperplanes.*

(1) *$\pi_1 X$ is a subgroup of a finitely generated right-angled Artin group, and hence a subgroup of $SL_n(\mathbb{Z})$ for some n.*
(2) *$\pi_1 X$ is residually torsion-free nilpotent.*
(3) *Let $Y \to X$ be a local-isometry of cube complexes with Y compact. Then $\pi_1 Y$ is a virtual retract of $\pi_1 X$.*
(4) *If $\pi_1 X$ is word-hyperbolic, then every quasiconvex subgroup of $\pi_1 X$ is separable.*

3.5. Action characterization

Let C be a special cube complex. Two hyperplanes A, B in C *intersect* if $A \cap B \neq \emptyset$. Two hyperplanes A, B *cross* if they intersect but are not equal. When a group G acts on C, we define $\mathrm{Intersector}_G(A, B) = \{g \in G\colon\ A \text{ intersects } gB\}$ and $\mathrm{Crosser}_G(A, B) = \{g \in G\colon\ A \text{ and } gB \text{ cross}\}$. We use $\mathrm{Stab}(A)$ to denote the usual $\mathrm{Stabilizer}(A)$ and we use $\mathrm{Stab}(\vec{A})$ to denote the subgroup that also stabilizes the two sides of A.

We say two oriented 2-sided hyperplanes $\vec{A}$, $\vec{B}$ *osculate* if there are oriented 1-cubes $\vec{a}$, $\vec{b}$ dual to $\vec{A}$, $\vec{B}$ such that $\vec{a}$, $\vec{b}$ have the same initial 0-cube or the same terminal 0-cube but do not lie in a 2-cube. Similarly, two unoriented hyperplane *osculate* if they have dual 1-cubes that share a common 0-cube but do not lie in a common 2-cube.

Define $\mathrm{Osculator}_G(\vec{A}, \vec{B}) = \{g \in G\colon\ \vec{A}, g\vec{B} \text{ Osculate}\}$, and define $\mathrm{Osculator}_G(A, B) = \{g \in G\colon\ A, gB \text{ Osculate}\}$. Note that $\mathrm{Osculator}_G(\vec{A}, \vec{B}) \subset \mathrm{Osculator}_G(A, B)$ for any choice of orientations.

Note that $\text{Osculator}_G(A, B)$, $\text{Crosser}_G(A, B)$ and $\text{Intersector}_G(A, B)$ are $\text{Stab}(A)$ left-invariant and $\text{Stab}(B)$ right-invariant. Similarly $\text{Osculator}_G(\vec{A}, \vec{B}) = \text{Stab}(\vec{A})\,\text{Osculator}_G(\vec{A}, \vec{B}) \times \text{Stab}(\vec{B})$.

Definition 3.4. We say G *acts specially* on the special cube complex C provided that

(1) $\text{Intersector}(A, A) = \text{Stab}(\vec{A})$ for each A.
(2) $\text{Osculator}_G(\vec{A}, \vec{A}) = \emptyset$ for each oriented hyperplane $\vec{A}$.
(3) $\text{Osculator}_G(A, B) = \emptyset$ for each pair of crossing hyperplanes A, B.

Theorem 3.5. *Let G act by combinatorial isometries on the special cube complex C, and let $\bar{C} = G\backslash C$ be the quotient. If G acts specially on C then $\bar{C}$ is special.*

Proof. The first condition implies that any element stabilizing a cube actually fixes it, and so the map $C \to \bar{C}$ is combinatorial.

The first condition implies that gA never crosses A and $\text{Stab}(A) = \text{Stab}(\vec{A})$. The first of these implies that for each hyperplane $\bar{A} = \text{Stab}(A)\backslash A$, the map $\bar{A} \to \bar{C}$ is an embedding. The second implies that the hyperplane $\bar{A}$ is 2-sided in $\bar{C}$.

The second condition implies that there is no self-osculation of a hyperplane in $\bar{C}$.

The third condition prevents an inter-osculation between hyperplanes in $\bar{C}$. □

Remark 3.6. The converse to Theorem 3.5 holds in the sense that if $\bar{C}$ arises from a group action with the property that any element stabilizing a cube actually fixes it, then if $\bar{C}$ is special then G acts specially.

4. Separable intersector criterion

In this section we obtain Theorem 4.1 which uses separability conditions to pass from a group acting on a nonpositively curved cube complex to a finite index subgroup that acts specially.

The special actions described in Section 3.5, were treated in [11] where we also gave a weaker version of Theorem 4.1 As the main objective of this paper is towards Coxeter groups, we are most interested in a special action on a CAT(0) cube complex – or in other words, on a simply-connected special cube complex. However, in Section 5 we give a sample application of the ideas applying special actions to cube complexes that are not simply-connected.

Theorem 4.1. *Let G act on a special cube complex C. Then G contains a finite index subgroup that acts specially provided the following conditions all hold*:

(1) *There are finitely many G orbits of hyperplanes.*
(2) *For each hyperplane A, there are finitely many* $\text{Stab}(A)$ *orbits of hyperplanes that osculate with A.*
(3) *For each hyperplane A, there are finitely many* $\text{Stab}(A)$ *orbits of hyperplanes that cross A.*
(4) *For each pair of intersecting hyperplanes A, B, the profinite closure of* $\text{Stab}(A)\,\text{Stab}(B)$ *is disjoint from* $\text{Osculator}_G(A, B)$.
(5) *For each hyperplane A, the profinite closure of* $\text{Stab}(A)$ *is disjoint from* $\text{Crosser}_G(A, A)$.

Proof. *Avoiding inter-osculating and self-osculating hyperplanes*: Let A, B be intersecting hyperplanes. Let $P(A,B)$ denote the profinite closure of $\mathrm{Stab}(A)\,\mathrm{Stab}(B)$ in G. Note that $P(A,B) = \mathrm{Stab}(A)P(A,B)\,\mathrm{Stab}(B)$. Indeed, for each $a, b \in \mathrm{Stab}(A), \mathrm{Stab}(B)$, the translate $aP(A,B)b$ is closed in the profinite topology, and $\mathrm{Stab}(A)\,\mathrm{Stab}(B) \subset aP(A,B)b$, so $P(A,B)$ must lie in $aP(A,B)b$.

Let $\mathrm{Osculates}(A)$ denote the set of hyperplanes that osculate with A. Note that $g\,\mathrm{Osculates}(A) = \mathrm{Osculates}(gA)$, and so $\mathrm{Osculates}(A) = \mathrm{Stab}(A)\,\mathrm{Osculates}(A)$.

Since $\mathrm{Osculates}(A)$ contains finitely many $\mathrm{Stab}(A)$-orbits, there is a finite set $J \subset G$, such that $\mathrm{Osculator}_G(A,B) = \mathrm{Stab}(A)J\,\mathrm{Stab}(B)$. By hypothesis, $\mathrm{Osculator}_G(A,B)$ and hence J is disjoint from $P(A,B)$ and so there is a finite index normal subgroup G_{AB} such that JG_{AB} is disjoint from $P(A,B)G_{AB}$.

Since $\mathrm{Stab}(A)\,\mathrm{Stab}(B) \subset P(A,B)$, we see that JG_{AB} is disjoint from $\mathrm{Stab}(A)\,\mathrm{Stab}(B)$, and so since G_{AB} is normal, $\mathrm{Stab}(A)J\,\mathrm{Stab}(B)$ is disjoint from G_{AB}, and we can conclude that $\mathrm{Osculator}_G(A,B)$ is disjoint from G_{AB}.

We follow the above procedure for a representative A_i of each G-orbit of hyperplane, and for a representative B_j of each $\mathrm{Stab}(A_i)$ orbit of hyperplane that intersects with A_i. We thus obtain finitely many finite index normal subgroups $G_{A_iB_j}$ whose intersection is a finite index normal subgroup F. Moreover, $F \cap \mathrm{Osculator}_G(A_i, B_j) = \emptyset$ for each A_i, B_j.

We now verify that $\mathrm{Osculator}_F(A,B) = \emptyset$ for each pair of intersecting hyperplanes A, B. Observe that $A = g_iA_i$ for some A_i and some $g_i \in G$, and so $B = g_ia_jB_j$ for some B_j intersecting A_i, and some $a_j \in \mathrm{Stab}(A_i)$. Let $f \in \mathrm{Osculator}_F(A,B)$. Then $f[g_i(a_jB_j)] \in \mathrm{Osculates}(g_iA_i)$. Left multiplying by $a_j^{-1}g_i^{-1}$ we find that:

$$a_j^{-1}g_i^{-1}fg_i(a_jB_j) \in \left(a_j^{-1}g_i^{-1}\right)\mathrm{Osculates}(g_iA_i) = \mathrm{Osculates}\left(\left(a_j^{-1}g_i^{-1}\right)g_iA_i\right) = \mathrm{Osculates}(A_i).$$

Since F is normal, $a_j^{-1}g_i^{-1}fg_ia_j = f' \in F$, and so $f'B_j \in \mathrm{Osculates}(A_i)$. Thus $f' \in \mathrm{Osculator}_G(A_i, B_j)$. But this contradicts that $F \cap \mathrm{Osculator}_G(A_i, B_j) = \emptyset$.

Hyperplanes embed in quotient: Let A be a hyperplane. By hypothesis, there are finitely many $\mathrm{Stab}(A)$ orbits of hyperplanes crossing A. Therefore $\mathrm{Crosser}_G(A,A) = \mathrm{Stab}(A)L\,\mathrm{Stab}(A)$ for some finite set $L \subset G$.

By hypothesis $L \cap P(A,A) = \emptyset$. We may therefore choose a finite index normal subgroup G_A such that $P(A,A)G_A \cap LG_A = \emptyset$.

Let A_i denote a complete set of representatives of the finitely many G-orbits of hyperplanes. For each i, let G_{A_i} be as above. Now let $K = \bigcap G_{A_i}$.

Then K has finite index in G, and $K \cap \mathrm{Crosser}_G(A,A) = \emptyset$ for each hyperplane A. Indeed, suppose A is a hyperplane, and suppose kA crosses A. Choose g, A_i such that $A = gA_i$, and note that A, kA cross if and only if gA_i, kgA_i cross if and only if A_i, $g^{-1}kgA_i$ cross. But K is normal, so $k' = g^{-1}kg \in K$, so $k' \in \mathrm{Crosser}_G(A_i, A_i)$ which is impossible.

Hyperplanes are 2-sided in quotient: Once we know that each hyperplane embeds in the quotient, we can show that there is a further finite index subgroup in which each hyperplane is 2-sided in the quotient. Indeed, if $A \subset K\backslash C$ is an embedded hyperplane, then there is an action of K on the Bass–Serre tree T corresponding to the associated splitting. The tree T has a bipartite structure, and we can choose an index $\leqslant 2$ subgroup K_A of K which preserves this bipartite structure. Then K_A will act without inversions on the edges, and so each translate of A in C maps to a 2-sided hyperplane in $K_A\backslash C$. We do this for each G-orbit representative A_i to obtain a finite index subgroup K_{A_i}.

We let $H = \bigcap_{A_i} K_{A_i}$. Then $H \cap F$ acts specially. □

Remark 4.2. The conditions in Theorem 4.1 are inherited by any finite index subgroup K of G. Thus if G satisfies these conditions, and is virtually torsion-free, then G has a finite index subgroup which is the fundamental group of a special cube complex.

Theorem 4.1 is aimed at nonpositively curved cube complexes with finitely many immersed hyperplanes. Its statement is substantially simplified in the presence of cocompactness as in the following statement comparable to what we proved in [11]:

Corollary 4.3. *Let G act on the special cube complex C. Then G has a finite index subgroup F that acts specially on C provided the following hold*:

(1) *G acts properly on C.*
(2) *G acts cocompactly on C.*
(3) $\text{Intersector}_G(A, B)$ *is separable for each pair of intersecting hyperplanes A, B.*
(4) $\text{Stab}_G(A)$ *is separable for each hyperplane.*

Proof. The finiteness conditions in the statement of Theorem 4.1 clearly hold when G also acts cocompactly. □

5. Virtual specialness of certain graphs of special cube complexes

Theorem 3.5 and Theorem 4.1 were crafted to study actions on a CAT(0) cube complex. We illustrate the wider applicability of actions on special cube complexes by proving the following result that is closely related to the results in [13]. Our later applications to Coxeter groups do not depend upon the results described here.

Hsu and Leary proved that if G splits as an HNN extension of an Artin group that conjugates one Artin subgroup to another (by a standard isomorphism), then G contains a finite index subgroup G' that also embeds in an Artin group. They ask whether a similar result holds for graphs of groups. Our result affirms this when the vertex groups are all right angled Artin groups.

Corollary 5.1. *Let H split as a finite graph of groups where each edge group and vertex group is a f.g. right-angled Artin group, and the embedding $G(\Lambda) \subset G(\Upsilon)$ of each edge group into a vertex group, is induced by embedding $\Lambda \subset \Upsilon$ as a full subgraph. Then H has a finite index subgroup that embeds in a right-angled Artin group.*

This follows immediately from the somewhat more general geometric restatement:

Theorem 5.2. *Let X decompose as a finite graph of spaces, where each vertex space X_v and edge space X_e is special with finitely many hyperplanes. Then X has a finite special cover provided the attaching maps of edge spaces satisfy the following*:

(1) *the attaching maps $X_e \to X_{\iota(e)}$ and $X_e \to X_{\tau(e)}$ are injective local-isometries*;
(2) *distinct hyperplanes of X_e map to distinct hyperplanes of $X_{\iota(e)}$ and $X_{\tau(e)}$*;
(3) *noncrossing hyperplanes map to noncrossing hyperplanes*;
(4) *no hyperplane of X_e extends in $X_{\iota(e)}$ to a hyperplane dual to an edge that intersects X_e in a single vertex* (*such a hyperplane of $X_{\iota(e)}$ is said to* inter-osculate X_e); *similarly no hyperplane of $X_{\tau(e)}$ inter-osculates X_e.*

The graph Γ of spaces decomposition means the following: There is a vertex space X_v and edge space X_e for each vertex and edge of Γ. And X is obtained from the disjoint union of the vertex spaces X_v and the thickened edge spaces $X_e \times [-1, 1]$ by gluing each subcomplex $X_e \times \{-1\}$ into $X_{\iota(e)}$ through the attaching map $X_e \to X_{\iota(e)}$, and similarly each subcomplex $X_e \times \{1\}$ into $X_{\tau(e)}$ through $X_e \to X_{\tau(e)}$. The attaching maps are assumed to be combinatorial (they send k-cubes to k-cubes) and are *local isometries* in the sense that the simplicial map induced on each vertex-link has full image. This insures that X is a nonpositively curved cube complex.

Proof of Theorem 5.2. Let $\widehat{X} \to X$ denote the covering space corresponding to the universal cover of the underlying graph Γ of X. The Galois group G of $\widehat{X}$ is thus the free group $\pi_1\Gamma$.

We first check that $\widehat{X}$ is special. As $\widehat{X}$ is a tree of special cube complexes, it is an increasing union of subcomplexes $\widehat{X}_n$, such that $\widehat{X}_0$ is a vertex space and $\widehat{X}_{n+1}$ is an edge of spaces, where one vertex space is $\widehat{X}_n$, the edge space $\widehat{X}_e$ is isomorphic to some X_e and the other vertex space $\widehat{X}_v$ is isomorphic to some vertex space X_v.

Any hyperplane pathology in $\widehat{X}$ would occur inside some X_n. To prove that X_{n+1} is special it suffices to prove that under the listed assumptions of the theorem an edge of special cube complexes is special. But in that case it is readily verifiable that the resulting cube complex has no self-intersection (by hyperplane-injectivity), no self-osculation (again by hyperplane-injectivity), no inter-osculation (by cross-injectivity and absence of hyperplanes interosculating the edge space). Also by hyperplane-injectivity every hyperplane is 2-sided.

Furthermore any complex mapping to either vertex space by a map satisfying the enumerated injectivity conditions of the theorem also maps to the full edge of spaces by a map enjoying the same injectivity conditions. This enables the continuation of the gluing procedure. By the previous argument, each vertex space is hyperplane-injective in $\widehat{X}$, and so the intersection with a vertex space of a hyperplane of $\widehat{X}$ is either empty or a single hyperplane.

It is clear that G acts properly on the special cube complex $\widehat{X}$. Let us now examine the finiteness properties of the hyperplanes of $\widehat{X}$.

Observe that each hyperplane A in $\widehat{X}$ projects to a tree $\bar{A}$ in $\widetilde{\Gamma}$. It is possible for distinct hyperplanes $A_1 \neq A_2$ to have $\bar{A}_1 = \bar{A}_2$, but we will regard these projections as distinct by keeping track of where their origins. The collection of hyperplanes $\{A_i\}$ in $\widehat{X}$ projects to a locally finite collection of trees $\{\bar{A}_i\}$ in $\widetilde{\Gamma}$. Indeed, since there are finitely many hyperplanes in each vertex space of X, we see that finitely many corresponding trees pass through any vertex of $\widetilde{\Gamma}$, and hence we obtain local finiteness since $\widehat{\Gamma}$ is itself locally finite.

The G-cocompactness of $\widetilde{\Gamma}$ combined with the local finiteness of $\{\bar{A}_i\}$ implies the various desired finiteness properties. Firstly, there are finitely many G-orbits of hyperplanes, and secondly, the stabilizer in G of each tree is cocompact since finitely many hyperplanes can have the exact same tree projection, we see that the stabilizer of each hyperplane acts cocompactly on its tree projection. These first two properties are readily verified from X itself, but our viewpoint enables us to use the local finiteness of $\bar{A}_i$ and the $\mathrm{Stab}(A)$-cocompactness of $\bar{A}$, to see the following: For each hyperplane A, and for each $K > 0$, there are finitely many $\mathrm{Stab}(A)$-orbits of trees $\bar{A}_i$ with $d(\bar{A}, \bar{A}_i) \leqslant K$. Setting $K = 2$, we see immediately that there are finitely many $\mathrm{Stab}(A)$-orbits of hyperplanes crossing or osculating A.

By [20], the double coset $H_1 g H_2$ is closed in the profinite topology for any finitely generated subgroups H_1, H_2 and element g. We can therefore apply Theorem 4.1 to conclude that G has a finite index subgroup J that acts specially on $\widehat{X}$. Thus $J \backslash \widehat{X}$ is a special cover of X by Theorem 3.5. $\square$

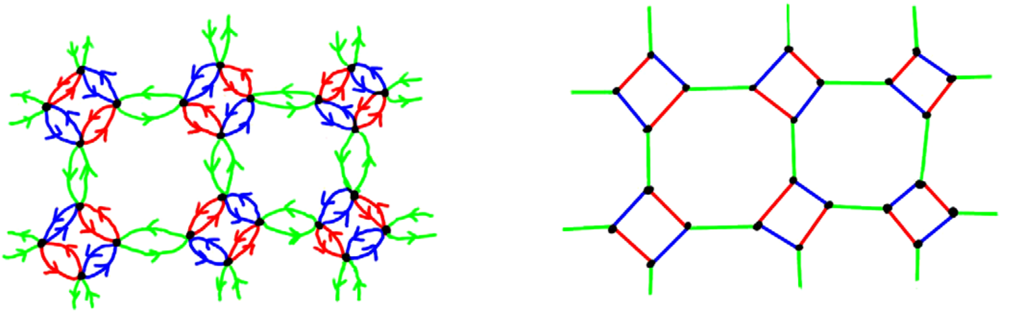

Fig. 2. $\widetilde{X} \to \bar{X}$ for $\langle a, b, c \mid a^2, b^2, c^2, (ab)^2, (bc)^4, (ac)^4 \rangle$.

6. The Niblo–Reeves cubulation of a Coxeter group

6.1. Coxeter group

Let $S = \{s_1, \ldots\}$ and let $M : S \times S \to \mathbb{N}^* \cup \infty$ be a symmetric function which also satisfies $M(s_i, s_j) = 1$ iff $i = j$. We use the notation $m_{ij} = M(s_i, s_j)$. The *Coxeter group* G associated with M is presented by:

$$\left\langle s_i, \ldots \;\middle|\; (s_i s_j)^{m_{ij}}\colon\, m_{ij} < \infty \right\rangle$$

Let X be the standard 2-complex of the above presentation, and let $\widetilde{X}$ be the universal cover. The Cayley graph $\Gamma(G, S)$ is the 1-skeleton of $\widetilde{X}$. Each generator of G actually has order 2, and each pair of generators generates a dihedral subgroup $\langle s_i, s_j \mid s_i^2, s_j^2, (s_i s_j)^{m_{ij}} \rangle$.

6.2. Coxeter complex

For each generator s_i, let X_i denote the standard 2-complex of $\langle s_i \mid s_i^2 \rangle$, and for each pair of generators s_i, s_j with $m_{ij} \neq \infty$, let X_{ij} denote the standard 2-complex of $\langle s_i, s_j \mid s_i^2, s_j^2, (s_i s_j)^{m_{ij}} \rangle$. We refer to the universal covers $\widetilde{X}_i$ and $\widetilde{X}_{ij}$ as a *bigon* and *dihedron* respectively. There are equivariant quotient maps $\widetilde{X}_i$ and $\widetilde{X}_{ij}$ to a line segment, and to a $2m_{ij}$-gon.

We modify $\widetilde{X}$ to obtain a convenient quotient polygonal complex $\bar{X}$ called the *Coxeter complex*. The complex $\bar{X}$ is obtained by identifying each *bigon* $\widetilde{X}_i$ contained in $\widetilde{X}$ to a single edge, and identifying each dihedron $\widetilde{X}_{ij}$ in $\widetilde{X}$ to a $2m_{ij}$-gon. See Fig. 2. There is a G-equivariant map $\widetilde{X} \to \bar{X}$.

6.3. The Davis–Moussong complex

There is a "thickening" of $\bar{X}$ into a CAT(0) space D, so that there is a G-equivariant map $\bar{X} \subset D$, and G acts cocompactly on D by isometries (see [5,17]). The walls of $\bar{X}$ that we shall discuss below extend to convex walls in D which facilitates their study.

6.4. The walls

A *wall* in $\bar{X}$ is a connected subspace W of $\bar{X}$ whose intersection with each 1-cell is either empty or the midpoint, and whose intersection with each 2-cell is either empty or a geodesic

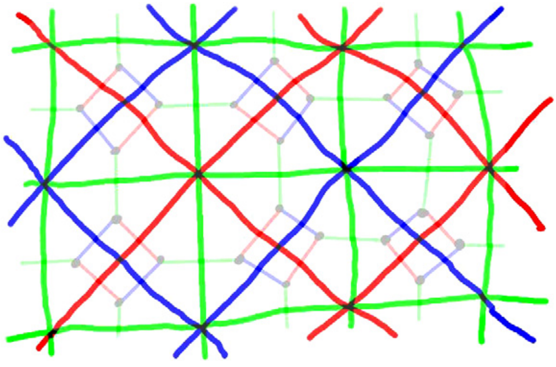

Fig. 3. Walls in the Coxeter complex of Fig. 2.

segment joining midpoints of 1-cells on opposite sides (see Fig. 3). Note that walls have an induced graphical structure.

Proposition 6.1. *Walls are well known to have several fundamental properties* [10]:

(1) *Each wall W separates $\bar{X}$ into two components.*
(2) *The midpoint of each 1-cell of $\bar{X}$ lies in a wall.*
(3) *The element s_i^g corresponding to a 1-cell acts on $\bar{X}$ by reflection along its associated wall.*
(4) $\mathrm{Stab}(W)$ *acts cocompactly on W.*
(5) *If W_a and W_b are walls, then the associated reflections generate a dihedral subgroup. If the walls are disjoint this subgroup is infinite-dihedral*; *if the walls cross at the center of an m_{ij}-gon it is isomorphic to D_r where r divides m_{ij}*; *and if the walls are equal it is isomorphic to $\mathbb{Z}_2$.*

6.5. Sageev's construction

Sageev's construction yields a group action of G on a CAT(0) cube complex C from the group action of G on the space with walls $\bar{X}$. We will certainly be relying on the details of his construction and refer the reader to [21,7,19,4,12] and especially [18] for the case of Coxeter groups. The hyperplanes of C are in one-to-one correspondence with the walls of $\bar{X}$. Moreover, the group action of G on the collection of walls is isomorphic to the group action of G on the collection of hyperplanes. In particular, the stabilizer of a hyperplane equals the stabilizer of the associated wall.

Hyperplanes of C cross if and only if the corresponding walls of $\bar{X}$ intersect transversally in some polygon of $\bar{X}$ (in which case we say the walls *cross*). And hyperplanes of C osculate if and only if the associated walls are disjoint and are not separated by a third wall (in which case we say the walls *osculate*).

6.6. The Niblo–Reeves cubulation

Niblo–Reeves deduced the finite-dimensionality of the cube complex C from the fact that there is a bound on the size of a collection of pairwise crossing walls.

They proved local-finiteness by showing that for a ball $B \subset \bar{X}$, there is an upper-bound on the cardinality of a set of walls $\{W_i\}$ such that no W_i separates B from W_j. This gives properness of the action, as G acts properly on $\bar{X}$.

By combining the results of Niblo–Reeves with a further result of Ben Williams [23], co-compactness holds when there are no Euclidean triangle Coxeter subgroups. This is proven by verifying that there are finitely many G-orbits of maximal collections of pairwise crossing walls, and hence finitely many G-orbits of maximal cubes in C.

An important ingredient in the understanding of walls in $\bar{X}$ was recently supplied by Pierre-Emmanuel Caprace, who proved the following result about the Coxeter complex [2, Theorem 4]: Here we define $d(W, W') + 1$ to equal the length of the shortest path which starts at a dual edge of W and ends at a dual edge of W'.

Proposition 6.2. *There exists a constant E such that if W, W' are walls in $\bar{X}$ with $d(W, W') \geqslant E$ then there is a third wall W'' that separates them.*

Remark 6.3 *(Finiteness Properties of Walls).* By construction each wall of $\bar{X}$ is a translate of a wall through an edge e_i at the origin, so that there are finitely many G-orbits of hyperplanes. Moreover, since $\mathrm{Stab}(W)$ acts cocompactly on W, and since $\bar{X}$ is locally finite, it is immediate that there are finitely many $\mathrm{Stab}(W)$ orbits of walls crossing W. Proposition 6.2 implies that any wall osculating with W satisfies $d(W, W') < E$. Consequently, there are finitely many $\mathrm{Stab}(W)$ orbits of walls osculating with W.

7. The separable subsets

A f.g. Coxeter group G has the following well-known properties (see for instance [16]):

Proposition 7.1.

(1) *G is linear and hence,*
(2) *G is residually finite and*
(3) *G is virtually torsion-free.*

Theorem 7.2. *Let A and B be walls in $\bar{X}$. Then*:

(1) $\mathrm{Stab}(A)$ *is separable in G.*
(2) $\mathrm{Intersector}(A, B)$ *is separable in G.*

Proof. Let r_a and r_b be the reflections in A and B. For a fixed n, consider the map:

$$\phi_n(g) = \left(r_a g r_b g^{-1}\right)^n.$$

Observe that $\phi_n : G \to G$ is continuous in the profinite topology since multiplication and inversion are continuous. Since G is residually finite, the trivial subgroup $\{1\}$ is closed in the profinite topology, and hence $\phi_n^{-1}(\{1\})$ is closed. We shall now use this observation to prove the theorem.

Observe that $\phi_1(g) = 1$ exactly when the reflection r_a equals the reflection $g r_b g^{-1}$, which is true exactly when $A = gB$. Thus when $A = B$, we have $\phi_1(g) = 1 \Leftrightarrow g \in \mathrm{Stab}(A)$, so $\mathrm{Stab}(A)$ is closed in the profinite topology.

By Proposition 6.1(5), the product $r_a(g^{-1}r_b g)$ has finite order exactly when the two reflection walls A, gB either equal or cross each other. Moreover, this element has order dividing some finite m_{ij}. Let $m = \text{LCM}\{m_{ij}\colon m_{ij} < \infty\}$. Then $\phi_m(g) = 1$ if and only if either $A = gB$ or A, gB cross. Thus $\phi_m^{-1}(\{1\}) = \text{Intersector}(A, B)$. □

8. Coxeter groups are virtually special

Theorem 8.1. *Let G be a finitely generated Coxeter group. Let C be the Niblo–Reeves* CAT(0) *cube complex of G. There exists a finite index torsion-free subgroup F of G such that $F\backslash C$ is a special cube complex.*

Proof. The finiteness properties indicated in Remark 6.3 together with the correspondence under Sageev's construction between walls and hyperplanes that is described in Section 6.5 imply the following three properties:

(1) There are finitely many G-orbits of walls in $\bar{X}$, and hence finitely many G-orbits of hyperplanes in C.
(2) For each wall A, there are finitely many Stab(A) orbits of walls that cross A, and hence the same statement holds for hyperplanes of C.
(3) For each wall A, there are finitely many Stab(A) orbits of walls that osculate with A, and hence the same statement holds for hyperplanes of C.

By Theorem 7.2, Stab(A) is separable and Intersector(B, A) is separable for each pair of walls A, B of $\bar{X}$, and hence the same holds for the associated hyperplanes of C.

We can therefore apply Theorem 4.1 to see that G contains a finite index subgroup F' which acts specially on C. By Proposition 7.1, G is virtually torsion-free, and so we can pass to a finite index torsion-free subgroup F of F' that acts both freely and specially. We conclude with Theorem 3.5. □

9. Problems

Problem 9.1. Following the notation and hypotheses of Theorem 8.1, what is the minimal index of $[G : F]$ such that $F\backslash C$ is special?

In the word-hyperbolic case the Coxeter group acts cocompactly on its Niblo–Reeves complex. However, while Euclidean triangle groups cannot act properly and cocompactly on a CAT(0) cube complex, it is conceivable that every Coxeter group has a finite index subgroup which acts properly and cocompactly on a CAT(0) cube complex. We pose the following:

Problem 9.2. Does every f.g. Coxeter group have a finite index subgroup that is the fundamental group of a nonpositively curved compact (special) cube complex?

We know that every subgroup that is quasiconvex with respect to the usual word metric is separable. While this is in agreement with the word-hyperbolic case, it seems too restrictive:

Problem 9.3. Is every quasi-isometrically embedded f.g. subgroup of a Coxeter group separable? Show that each subgroup that is quasiconvex with respect to the CAT(0) metric of the Davis–Moussong complex is separable.

We close with a problem which appears to be deeper than the ones above. Perhaps it is a bit premature, since even cubulating Artin groups would solve many of the outstanding problems related to them:

Problem 9.4. Does every finitely generated Artin group contain a finite index subgroup that is the fundamental group of a special cube complex with finitely many hyperplanes?

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