QUATERNION ALGEBRAS and INVARIANTS of VIRTUAL KNOTS and LINKS II: The Hyperbolic Case

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ABSTRACT

Let A, B be invertible, non-commuting elements of a ring R. Suppose that $A-1$ is also invertible and that the equation

$$[B,(A-1)(A,B)]=0$$

called the fundamental equation is satisfied. Then an invariant R-module is defined for any diagram of a (virtual) knot or link. Solutions in the classic quaternion case have been found by Bartholomew, Budden and Fenn. Solutions in the generalised quaternion case have been found by Fenn in an earlier paper. These latter solutions are only partial in the case of 2×2 matrices and the aim of this paper is to provide solutions to the missing cases.

1 Introduction

Let A, B be invertible, non-commuting elements of a ring R. Suppose that $A-1$ is also invertible. Our aim is to find solutions to the equation

$$A^{-1}B^{-1}AB-B^{-1}AB=BA^{-1}B^{-1}A-A,$$

called the *fundamental equation*. In this case an invariant R-module is defined for any diagram of a (virtual) knot or link. Solutions in the classic quaternion case have been found, see [**BF**] , [**BuF**] . Solutions in the generalised quaternion case have been found, see [**F**] . These are only partial in the case of 2×2 matrices; that is sufficient but not necessary conditions were given. The matrices satisfying these conditions were called *matching*. The aim of this paper is to provide solutions of the missing matrices: the mismatching or hyperbolic matrices. This means that this paper together with earlier papers provides all 2×2 matrix solutions to the fundamental equation. These solutions may be summed up with the help of the following theorem.

Theorem 1.1 Suppose A,B are two non-commuting 2×2 matrix solutions of the fundamental equation. Then either

$$\mathrm{tr}(A)=\det(A) \text{ and } A\mathrm{adj}(B)+B\mathrm{adj}(A)=0$$

or the pair A,B are similar to a pair of the form

$$A=\begin{pmatrix} a_0+a_3 & 2a_1 \\ 0 & a_0-a_3\end{pmatrix}\qquad B=\begin{pmatrix} \frac{2b_3}{a_0-a_3} & 2b_1 \\ 0 & 2b_3(\frac{1}{a_0-a_3}-2)\end{pmatrix}$$

There are a number of sporadic $n \times n$ matrix solutions to the fundamental equation which will appear in a further paper with V. Turaev. However these are almost certainly not complete and finding the general $n \times n$ solution is probably very hard.

With these solutions whole new families of invariant modules and polynomials of virtual knots and links are defined. An appendix where these are calculated from the examples in the table of N. Kamada will be put on the web.

Many of the conventions and notation can be found in **[F]** . We will reproduce the details necessary for the understanding of this paper and leave fine details for the interested reader in **[F]** .

2 Generalised Quaternions

Let F be a field of characteristic not equal to 2. Pick two non-zero elements $\lambda,\ \mu$ in F. Let $\left(\frac{\lambda,\,\mu}{F}\right)$ denote the algebra of dimension 4 over F with basis $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and relations $\mathbf{i}^2 = \lambda,\ \mathbf{j}^2 = \mu,\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}$. The multiplication table is given by

$$\begin{array}{c} \\ \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{array}\begin{array}{c} \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{array} \\ \begin{pmatrix} \lambda & \mathbf{k} & \lambda\mathbf{j} \\ -\mathbf{k} & \mu & -\mu\mathbf{i} \\ -\lambda\mathbf{j} & \mu\mathbf{i} & -\lambda\mu \end{pmatrix} \end{array}.$$

Throughout the paper a general quaternion algebra will be denoted by $\mathcal{Q}$. Elements of $\mathcal{Q}$ are called (generalized) quaternions. The field F is called the **underlying** field and the elements $\lambda\ \mu$ the **parameters** of the algebra. We will denote quaternions by capital roman letters such as $A, B, \ldots$ and (if pure) by bold face lower case, $\mathbf{a}, \mathbf{b}, \ldots$. Field elements, (scalars) will be denoted by lower case roman letters such as $a, b, \ldots$ and lower case greek letters such as $\alpha, \beta, \ldots$.

The classical quaternions are $\left(\frac{-1,\,-1}{\mathbb{R}}\right)$. The algebra of 2×2 matrices with entries in F is $M_2(F) = \left(\frac{-1,\,1}{F}\right)$.

2.1 Conjugation, Norm and Trace

Let $A = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ be a quaternion where $a_0, a_1, a_2, a_3 \in F$. The coordinate a_0 is called the **scalar** part of A and the 3-vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is called the **pure** part of A. Evidently $A = a_0 + \mathbf{a}$ is the sum of its scalar and pure parts and is pure if its scalar part is zero and is a scalar if its pure part is zero.

The **conjugate** of A is $\overline{A} = a_0 - \mathbf{a}$, the **norm** of A is $N(A) = A\overline{A}$ and the **trace** of A is $\mathrm{tr}(A) = A + \overline{A}$.

Conjugation is an anti-isomorphism of order 2. That is it satisfies

$$\overline{A+B} = \overline{A} + \overline{B}, \quad \overline{AB} = \overline{B}\,\overline{A}, \quad \overline{aA} = a\overline{A}, \quad \overline{\overline{A}} = A.$$

Also $\overline{A} = A$ if and only if A is a scalar and $\overline{A} = -A$ if and only if A is pure.

The norm is a scalar satisfying $N(AB) = N(A)N(B)$. We will denote the set of values of the norm function by $\mathcal{N}$. It is a multiplicatively closed subset of F and $\mathcal{N}^* = \mathcal{N} - \{0\}$ is a multiplicative subgroup of F^*. An element A has an inverse if and only if $N(A) \neq 0$ in which case $A^{-1} = N(A)^{-1}\overline{A}$.

The trace of a quaternion is twice its scalar part.

2.2 Multiplying Quaternions

Let A, B be two quaternions.There is a bilinear form given by

$$A \cdot B = \frac{1}{2}(A\overline{B} + B\overline{A}) = \frac{1}{2}(\overline{A}B + \overline{B}A) = \frac{1}{2}\mathrm{tr}(A\overline{B}).$$

In terms of coordinates this is

$$A \cdot B = a_0b_0 - \lambda a_1b_1 - \mu a_2b_2 + \lambda\mu a_3b_3.$$

Since λ and μ are non-zero this is a non-degenerate form. The corresponding quadratic form is

$$N(A) = a_0^2 - \lambda a_1^2 - \mu a_2^2 + \lambda\mu a_3^2.$$

Let $\mathbf{a}, \mathbf{b}$ be pure quaternions. Then

$$\mathbf{ab} = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

where

$$\mathbf{a} \cdot \mathbf{b} = -\lambda a_1b_1 - \mu a_2b_2 + \lambda\mu a_3b_3$$

is the restriction of the bilinear form to the pure quaternions and $\mathbf{a} \times \mathbf{b}$ is the **cross product** defined symbolically by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} -\mu\mathbf{i} & -\lambda\mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The cross product has the usual rules of bilinearity and skew symmetry. The triple cross product expansion

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$$

is easily verified. The **scalar triple product** is

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \lambda\mu \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

from which all the usual rules (except volume) can be deduced.

2.3 Dependancy Criteria

In this subsection we will consider conditions for sets of quaternions to be linearly dependant or otherwise. A non-zero element, A, of $\mathcal{Q}$ is called **isotropic** if $N(A) = 0$ and **anisotropic** otherwise. So only non-zero anisotropic elements have inverses. We note the following theorem.

Theorem 2.2 The following statements about a quaternion algebra $\mathcal{Q}$ are equivalent.

1. $\mathcal{Q}$ contains an isotropic element.
2. $\mathcal{Q}$ is the sum of two hyperbolic planes.
3. $\mathcal{Q}$ is not a division algebra.
4. $\mathcal{Q}$ is $M_2(F)$.

Proof See [**L**] p 58.

We will call a quaternion algebra above **hyperbolic**. Otherwise it is called **anisotropic**. The classic quaternions are anisotropic: 2×2 matrices are hyperbolic.

Lemma 2.3

A pair of pure quaternions $\mathbf{a}, \mathbf{b}$ is linearly dependant if and only if $\mathbf{a} \times \mathbf{b} = 0$.

Proof The proof is clear one way using the antisymmetry of the cross product. Conversely suppose $\mathbf{a} \times \mathbf{b} = 0$. Then $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} = 0$. This can be made into a linear dependancy by a suitable choice of $\mathbf{c}$, for example if $\mathbf{a} \cdot \mathbf{c} \neq 0$. □

As a corollary we have the following

Lemma 2.4

Two quaternions commute if and only their pure parts are linearly dependant. □

Now we look for conditions for the triple of pure quaternions, $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$, to be linearly dependant. The required condition is given by the following lemma.

Lemma 2.5

The pure quaternions $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$, are linearly dependant if and only if

$$N(\mathbf{a})N(\mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})^2.$$

This is equivalent to the equations

$$N(\mathbf{a} \times \mathbf{b}) = -\mu(a_2 b_3 - a_3 b_2)^2 - \lambda(a_1 b_3 - a_3 b_1)^2 + (a_1 b_2 - a_2 b_1)^2 = 0,$$

ie $\mathbf{a} \times \mathbf{b}$ is isotropic or zero.

Proof Three 3-dimensional vectors are linearly dependant if and only if the determinant they form by rows is zero. In the case of pure quaternions this means the scalar triple product is zero

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \lambda\mu \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Replacing $\mathbf{c}$ with $\mathbf{a} \times \mathbf{b}$ and expanding out using the triple cross product formula gives the first equation. Using the expansion formulæ

$$N(\mathbf{a} \times \mathbf{b}) = N(\mathbf{a})N(\mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2$$

gives the second formula. □

We have the following corollary.

Lemma 2.6

If $\mathbf{a}, \mathbf{b}$ are linearly independant pure quaternions and $\mathbf{a} \times \mathbf{b}$ is anisotropic, then the triple $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$, is linearly independant. □

2.4 2×2 matrices

We will interpret all the previous results in terms of 2×2 matrices, $M_2(F) = \left(\frac{-1,\, 1}{F}\right)$ or $\left(\frac{1,\, 1}{F}\right)$. This is the only quaternion algebra with zero divisors.

The generators of $\left(\frac{-1,\, 1}{F}\right)$ are, together with the identity, the Pauli matrices

$$\mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; \mathbf{j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \mathbf{k} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By an abuse of notation we will often confuse the scalar matrix $\begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix}$ with the corresponding field element ν.

A general matrix can be written uniquely as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{2}\left[(\alpha+\delta) + (\beta-\gamma)\mathbf{i} + (\beta+\gamma)\mathbf{j} + (\alpha-\delta)\mathbf{k}\right]$$

Conversely

$$A = a_0 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = \begin{pmatrix} a_0 + a_3 & a_2 + a_1 \\ a_2 - a_1 & a_0 - a_3 \end{pmatrix}$$

Conjugation is

$$\overline{A} = \text{adj}A = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} a_0 - a_3 & -a_2 - a_1 \\ a_1 - a_2 & a_0 + a_3 \end{pmatrix}$$

and norm is

$$N(A) = \det A = \alpha\delta - \beta\gamma = a_0^2 + a_1^2 - a_2^2 - a_3^2$$

The scalar part of A is $a_0 = \text{tr}A/2 = (\alpha+\delta)/2$ and the pure part is

$$\begin{pmatrix} a_3 & a_2 + a_1 \\ a_2 - a_1 & -a_3 \end{pmatrix} = \begin{pmatrix} (\alpha-\delta)/2 & \beta \\ \gamma & (\delta-\alpha)/2 \end{pmatrix}$$

2.5 Multiplying Matrices

Lemma 2.2

Suppose $A, B \in M_2(F) = \left(\frac{-1,\,1}{F}\right)$. Then $AB = AB$.

The statement is deliberately provocative. It says that multiplying A, B as matrices and as quaternions is the same. This can be checked directly. □

The above lemma allows quick checking of formulæ so if

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \text{ and } B = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \text{ then } A \cdot B = \frac{1}{2}(\alpha_1\beta_4 - \alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_4\beta_1)$$

If $\mathbf{a} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & -\alpha_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & -\beta_1 \end{pmatrix}$ are pure then $\mathbf{a}\cdot\mathbf{b} = -\alpha_1\beta_1 - (\alpha_2\beta_3 + \alpha_3\beta_2)/2$ and

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (\alpha_2\beta_3 - \alpha_3\beta_2)/2 & \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 & (\alpha_3\beta_2 - \alpha_2\beta_3)/2 \end{pmatrix}$$

3 Solving the Fundamental Equation

Given a set X let S be an endomorphism of X^2. Such an S is called a **switch** if

1 S is invertible and

2 the set theoretic Yang-Baxter equation

$$(S \times id)(id \times S)(S \times id) = (id \times S)(S \times id)(id \times S)$$

is satisfied. Switches are used in **[FJK]** to define biracks and biquandles by the formula

$$S(a, b) = (b_a, a^b).$$

Switches can be used to find representations of the virtual braid groups and invariants of virtual knots and links, see **[FJK]** , **[BF]** , **[BuF]** and **[F]** .

We are looking for linear solutions. That is $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where the matrix entries A, B, C, D are elements of R, an associative but not necessarily commutative ring and X is a left R-module.

The solutions can be divided into two types when the entries are not zero divisors.

The commutative case

0 : The identity

1 : $$S = \begin{pmatrix} 0 & B \\ C & 1 - BC \end{pmatrix} \text{ or } S = \begin{pmatrix} 1 - BC & B \\ C & 0 \end{pmatrix}$$

where B and C are arbitary commuting invertible elements.

The non-commutative case

2 : $$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A, A - 1, B$ are invertible, A, B do not commute and satisfy the fundamental equation

$$A^{-1}B^{-1}AB - BA^{-1}B^{-1}A = B^{-1}AB - A$$

moreover

$$C = A^{-1}B^{-1}A(1 - A),\ D = 1 - A^{-1}B^{-1}AB.$$

There are also similar solutions where A, D and B, C are interchanged. We are only interested in this last case and are therefore looking for generalised quaternions $A = a_0 + \mathbf{a}$ and $B = b_0 + \mathbf{b}$ s.t.

$$[B, (A-1)A^{-1}B^{-1}AB] = 0$$

Since A and B do not commute $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. As in **[BuF]** and **[F]** the linear relation

$$\begin{aligned}(\mathrm{tr}(A) - N(A))N(\mathbf{b})\mathbf{a} + (N(A) - \mathrm{tr}(A))(\mathbf{a}\cdot\mathbf{b})\mathbf{b} \\ + (b_0(N(A) - \mathrm{tr}(A)) + 2A\cdot B)\mathbf{a}\times\mathbf{b} = 0 \end{aligned} \qquad (1)$$

holds.

The paper **[F]** has solved the **matching solutions**. That is solutions where $N(A) = \mathrm{tr}(A)$ and $A \cdot B = 0$. So we are interested in the **mismatching solutions**. In this case A, B satisfy the fundamental equation and $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ are linearly dependent.

4 Finding Linearly Dependent Triples $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$

In this section we find precise conditions for the triple $\mathbf{a}, \mathbf{b}$ and $\mathbf{a}\times\mathbf{b}$ to be linearly dependant. Recall that this happens if $N(\mathbf{a}\times\mathbf{b}) = 0$, ie $\mathbf{a}\times\mathbf{b}$ is isotropic. A vector is isotropic if it lies in X, the right circular cone $x_1^2 - x_2^2 - x_3^2 = 0$.

Let

$$\mathbf{a}\cdot_{\mathbb{E}}\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

and

$$\mathbf{a}\times_{\mathbb{E}}\mathbf{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

denote "euclidean" scalar and cross product respectively. This is to distinguish them from the "hyperbolic" versions

$$a\cdot_{\mathbb{H}}\mathbf{b} = a_1b_1 - a_2b_2 - a_3b_3$$

and

$$\mathbf{a}\times_{\mathbb{H}}\mathbf{b} = \begin{vmatrix} -i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Let ρ be the involution given by $\rho(x_1, x_2, x_3) = (-x_1, x_2, x_3)$. Then X is invariant under ρ and $\rho(\mathbf{a}\times_{\mathbb{H}}\mathbf{b}) = \mathbf{a}\times_{\mathbb{E}}\mathbf{b}$.

If $\mathbf{c}$ is isotropic let $\mathbf{a}, \mathbf{b}$ lie in the plane $\mathbf{c}\cdot_{\mathbb{E}}\mathbf{x} = 0$. This plane meets the cone in the generator containing $\rho(\mathbf{c})$. So $\mathbf{a}\times_{\mathbb{E}}\mathbf{b}$ is parallel to $\mathbf{c}$ and $\mathbf{a}\times_{\mathbb{H}}\mathbf{b}$ is isotropic. This means that the triple $\mathbf{a}, \mathbf{b}, \mathbf{a} \times_{\mathbb{H}} \mathbf{b}$ is linearly dependant. Moreover all examples of such triples are obtained in this way.

From now on $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ will have their original (hyperbolic) meanings.

4.1 A Worked Example

Now we find a generic family of triples containing all the properties needed. This is summed up by the following theorem

Theorem 4.2 Without loss of generality we can assume that $\mathbf{a}$ and $\mathbf{b}$ lie in the plane $x_1 - x_2 = 0$. The most general examples being

$$\mathbf{a} = \begin{pmatrix} a_3 & 2a_1 \\ 0 & -a_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_3 & 2b_1 \\ 0 & -b_3 \end{pmatrix}, \quad \mathbf{a} \times \mathbf{b} = \begin{pmatrix} 0 & 2(a_3 b_1 - a_1 b_3) \\ 0 & 0 \end{pmatrix}.$$

$$\text{So } b_3\mathbf{a} - a_3\mathbf{b} + \mathbf{a} \times \mathbf{b} = 0.$$

Proof Some small lemmas are needed.

Lemma 4.3

For any pure quaternions $\mathbf{a}, \mathbf{b}, \mathbf{c}$, we have

$$(\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{c}) = N(\mathbf{c})(\mathbf{a} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{c})$$

Proof This is just a routine calculation. □

We will use conjugation in the group theoretic sense, (ie. A conjugated by B is $B^{-1}AB$). Since the word conjugation is already being used in a rather different sense (analogous to complex conjugation) we will use the term *group-conjugation*. Note that the set of solutions to the fundamental equation is invariant under group-conjugation.

Lemma 4.4

The inner product is invariant under group-conjugation. This means that for any quaternion C and any pure quaternions $\mathbf{a}$ and $\mathbf{b}$ we have

$$C^{-1}\mathbf{a}C \cdot C^{-1}\mathbf{b}C = \mathbf{a} \cdot \mathbf{b}$$

Proof Tedious but routine calculation using the above lemma. □

Lemma 4.5

Any pure quaternion is group-conjugate to a pure quaternion of the form $a_1\mathbf{i} + a_2\mathbf{j}$. In particular any isotropic pure quaternion is group-conjugate to one of the form $x(\mathbf{i} + \mathbf{j}),\ x \in F$

Proof If $a_3 = 0$ then we are already there, so we will assume otherwise.

Case 1: $a_2^2 + a_3^2 \neq 0$.

Consider $C = (-a_2 + \sqrt{a_2^2 + a_3^2}) - a_3\mathbf{i}$. Then C is invertible whenever -

$$N(C) = 2(a_2^2 + a_3^2 - a_2\sqrt{a_2^2 + a_3^2}) = 2(\sqrt{a_2^2 + a_3^2})(\sqrt{a_2^2 + a_3^2} - a_2) \neq 0$$

Hence C is invertible. Moreover

$$C^{-1}\mathbf{a}C = a_1\mathbf{i} + \sqrt{a_2^2 + a_3^2}\ \mathbf{j}$$

so we are done.

Case 2: $a_2^2 + a_3^2 = 0$ Again a_2 and a_3 are not zero and so the underlying field must have a square root of -1, unique up to multiplication by -1, which we will call I. In fact $I = \pm a_3/a_2$. Group-conjugating $\mathbf{a}$ by $1 + I\mathbf{i}$ gives $a_1\mathbf{i}$, as required. □

Isotropic vectors are invariant under group conjugation and so by the above we can assume an isotropic vector is of the form $x(\mathbf{i} + \mathbf{j})$. Hence from above $\mathbf{a}$ and $\mathbf{b}$ can be conjugated such that,

$$a_1(\mathbf{i} - \mathbf{j}) \cdot_{\mathbb{E}} \mathbf{a} = a_1(\mathbf{i} - \mathbf{j}) \cdot_{\mathbb{E}} \mathbf{b} = 0.$$

The most general case then is (up to group-conjugation) $\mathbf{a} = a_1\mathbf{i} + a_1\mathbf{j} + a_3\mathbf{k}$ $\mathbf{b} = b_1\mathbf{i} + b_1\mathbf{j} + b_3\mathbf{k}$. Substituting

$$\mathbf{i} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \mathbf{j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \mathbf{k} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

concludes the proof of Theorem 4.2 □

5 How To Find A and B given Linearly Dependent $\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

In this section we complete our theoretical solution of $\mathbf{a}, \mathbf{b}$ to find A and B.

Assume that $\mathbf{a}, \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ are linearly dependent. That is for some coefficients not all zero there is a linear relationship

$$\lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{a} \times \mathbf{b} = 0$$

We are assuming that A and B do not commute, so $\mathbf{a}, \mathbf{b}$ are linearly independant and hence $\lambda_3 \neq 0$ and we can write

$$\lambda_1 \mathbf{a} + \lambda_2 \mathbf{b} + \mathbf{a} \times \mathbf{b} = 0 \tag{2}$$

where λ_1 and λ_2 are unique. Both λ_1 and λ_2 cannot be zero for then by our earlier discussion A and B would commute.

We can obtain information about λ_1 and λ_2 by taking the cross product of $\mathbf{a}$ with $\mathbf{a} \times \mathbf{b}$

$$\begin{aligned}
\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) &= -\lambda_1 \mathbf{a} \times \mathbf{a} - \lambda_2 \mathbf{a} \times \mathbf{b} \\
&= \lambda_1 \lambda_2 \mathbf{a} + \lambda_2^2 \mathbf{b} \text{ on the one hand, and} \\
&= (\mathbf{b} \cdot \mathbf{a})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b} \text{ by standard expansion rules}
\end{aligned}$$

Comparing coefficients, we have -

$$\begin{aligned}
\lambda_1 \lambda_2 &= \mathbf{a} \cdot \mathbf{b}; \\
\lambda_2^2 &= -N(\mathbf{a}) : \text{ and similarly} \\
\lambda_1^2 &= -N(\mathbf{b})
\end{aligned} \tag{3}$$

Comparing (2) and (3) with (1) and separating coefficients we get the two equations

$$\lambda_i [(N(A) - tr(A))\lambda_1 + b_0 N(A) + 2\lambda_1 \lambda_2] = 0 \qquad i = 1,\ 2$$

Since we cannot have both λ_1 and $\lambda_2 = 0$, else we get a commuting solution, we have -

$$\begin{aligned}
b_0 &= \frac{(tr(A) - N(A) - 2\lambda_2)\lambda_1}{N(A)} \\
&= \lambda_1 \left(\frac{2}{a_0 + \lambda_2} - 1 \right) \\
&= \sqrt{-N(\mathbf{b})} \left(\frac{2}{a_0 + \sqrt{-N(\mathbf{a})}} - 1 \right)
\end{aligned} \tag{4}$$

Thus, if $\mathbf{a}$ and $\mathbf{b}$ are such that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ are linearly dependent, then we can pick any a_0, not equal to $-\sqrt{-N(\mathbf{a})}$, then choose b_0 according to (4), and we will have A and B that satisfy the fundamental equation. Note that the roots will have opposite sign.

To be a switch we also require B, A, $A - I$ and S to be invertible.

Now A is singular if and only if $N(A) = a_0^2 - \lambda_2^2 = 0$ if and only if $a_0 = \pm\lambda_2$

$(A - 1)$ is singular if and only if $a_0 = 1 \pm \lambda_2$

and B is singular if and only if $b_0 = \pm\lambda_2$ if and only if $\lambda_1 = 0$ or $a_0 = 1 - \lambda_2$.

Lemma 5.6

If A, B and $(A - 1)$ are invertible then so is S

Proof According to **[BF]** , S is invertible if $\Delta' = C^{-1}D - A^{-1}B$ is invertible. Using $C == A^{-1}B^{-1}A(1-A),\ D = 1-A^{-1}B^{-1}AB$ we find that $\Delta' = (1-A)^{-1}A^{-1}B(A-1)$. $\square$

So we require $\lambda_1 \neq 0$ and $a_0 \neq \pm\lambda_2, 1 \pm \lambda_2$.

5.1 The Worked Example(continued)

We have $\lambda_1 = b_3$ and $\lambda_2 = -a_3$. So $b_0 = b_3(\frac{2}{a_0 - a_3} - 1)$

hence all mismatching solutions are conjugate to ones of the form -

$$A = \begin{pmatrix} a_0 + a_3 & 2a_1 \\ 0 & a_0 - a_3 \end{pmatrix} \qquad B = \begin{pmatrix} \frac{2b_3}{a_0 - a_3} & 2b_1 \\ 0 & 2b_3(\frac{1}{a_0 - a_3} - 2) \end{pmatrix}$$

The matrices A and $A - I$ will be invertible as long as $a_0 \neq \pm a_3,\ 2 + a_3\ 1 \pm a_3$.

6 References

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