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**QUATERNION ALGEBRAS and
INVARIANTS of VIRTUAL KNOTS and LINKS
I: The Elliptic Case**

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ABSTRACT

In this paper we show how generalized quaternions including some 2×2 matrices can be used to find solutions of the equation

$$[B, (A - 1)(A, B)] = 0.$$

These solutions can then be used to find polynomial invariants of virtual knots and links. The remaining 2×2 matrices will be considered in a later paper.

1 Introduction

In this paper and a subsequent paper, jointly written with S. Budden, we will find all solutions of the equation

$$A^{-1}B^{-1}AB - B^{-1}AB = BA^{-1}B^{-1}A - A$$

in which A, B are elements of a quaternion algebra. This includes the case when A, B are 2×2 matrices. This later case can be divided into two cases, the elliptic case considered here and the hyperbolic case considered in the next paper.

Note that the above equation can also be written

$$[B, (A - 1)(A, B)] = 0$$

where $[X, Y] = XY - YX$ is one type of commutator and $(X, Y) = X^{-1}Y^{-1}XY$ is another.

Consider the algebra with the following presentation

$$\mathcal{F} = \{A, B \mid A^{-1}B^{-1}AB - B^{-1}AB = BA^{-1}B^{-1}A - A\}.$$

In this paper we will call this the **fundamental** algebra and the single relation will be called the fundamental relation or equation. The aim of this and the second paper is to find all representations of this algebra into quaternion algebras.

The fundamental relation arises naturally from attempts to find representations of the braid group. Representations of the fundamental algebra as matrices can be used to define representations of the virtual braid group and invariants of virtual knots and links. In **[BuF]** we found

a complete set of conditions for two classic quaternions, A, B to be solutions of the fundamental equation. In this paper this result is generalised to give necessary and sufficient conditions for generalized quaternions to satisfy the fundamental relation, except in the case of all 2×2 matrices where only sufficient conditions are given. The remaining case will be considered in the next paper.

Let K be a virtual knot or link. We show that if A, B are two elements of a ring, R , satisfying the fundamental relation, then there a left R -module \mathcal{M} , dependant on A, B which is an invariant of K . From this module more tractible invariants such as polynomials can be defined.

As an application, we define two 4-variable polynomials of virtual knots and links. In addition, we give conclusive proof of the fact, only hinted at in earlier papers, that invariants defined in this manner extend the Alexander polynomial for classical knots and links.

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2 The fundamental equation and its justification

Given a set X let S be an endomorphism of X^2 . In [FJK] , such an S is called a **switch** if

- 1 S is invertible and
- 2 the set theoretic Yang-Baxter equation

$$(S \times id)(id \times S)(S \times id) = (id \times S)(S \times id)(id \times S)$$

is satisfied. Switches are used in [FJK] to define biracks and biquandles by the formula

$$S(a, b) = (b_a, a^b).$$

Given a switch S there is a representation of the braid group B_n into the group of permutations of X^n defined by

$$\sigma_i \longrightarrow (id)^{i-1} \times S \times (id)^{n-i-1}$$

where σ_i are the standard generators. Denote this representation by $\rho = \rho(S, n)$.

In this paper we will only be interested in the following linear switches. So let R be an associative but not necessarily commutative ring and let X be a left R -module. Suppose

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where the matrix entries A, B, C, D are elements of R . The 3×3 matrices of the Yang-Baxter equation are

$$S \times id = \begin{pmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad id \times S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & C & D \end{pmatrix}$$

The representation ρ is now into $n \times n$ matrices with entries from R .

Let us consider methods to find such switches S . It is not difficult to see that the following 7 equations are necessary and sufficient conditions for an invertible S to be a switch,

$$\begin{aligned} 1 : A &= A^2 + BAC & 2 : [B, A] &= BAD \\ 3 : [C, D] &= CDA & 4 : D &= D^2 + CDB \\ 5 : [A, C] &= DAC & 6 : [D, B] &= ADB \\ 7 : [C, B] &= ADA - DAD \end{aligned}$$

where $[X, Y] = XY - YX$.

Examples of switches are

0 : The identity

$$1 : S = \begin{pmatrix} 0 & B \\ C & 1 - BC \end{pmatrix} \text{ or } S = \begin{pmatrix} 1 - BC & B \\ C & 0 \end{pmatrix}$$

where B and C are arbitrary commuting invertible elements. This is called the **Alexander** switch. A special case of this, when $B = 1$, is called the **Bureau** switch.

$$2 : S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A, A - 1, B$ are invertible, A, B do not commute and satisfy the fundamental equation

$$A^{-1}B^{-1}AB - BA^{-1}B^{-1}A = B^{-1}AB - A$$

moreover

$$C = A^{-1}B^{-1}A(1 - A), \quad D = 1 - A^{-1}B^{-1}AB.$$

We will call this the **non-commuting** switch. A special case of this is the matrix with quaternion entries

$$S = \begin{pmatrix} 1 + i & j \\ -j & 1 + i \end{pmatrix}$$

called the **Budapest** switch.

If $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a switch then so is $S(t) = \begin{pmatrix} A & tB \\ t^{-1}C & D \end{pmatrix}$ where t is a commuting variable. We say that $S(t)$ is S **augmented** by t .

In [BuF] and [BF] the following results can be found.

Theorem 2.1 *Suppose R is a division ring. Then any switch is one of the examples above.*

□

Of course other types are possible, see [Cs] in which divisors of zero are used.

The representation of the braid group induced by any non-commuting switch looks complicated but is in fact equivalent to the Burau representation. This has been pointed out previously by Dehornoy, see [De] . The following lemma gives an explicit proof.

Lemma 2.2 Let $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a non-commuting switch and let $S' = \begin{pmatrix} 0 & 1 \\ Q & 1-Q \end{pmatrix}$ be the Burau switch where $Q = (1-A)(1-D)$. Let M be the $n \times n$ matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ A & B & 0 & \cdots & 0 \\ A & BA & B^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A & BA & B^2A & \cdots & B^{n-1} \end{pmatrix}$$

In words: the rows of M , after the first, start with A and then the previous row multiplied on the left by B . Clearly M is invertible. Then $\rho(S, n) = M^{-1}\rho(S', n)M$.

Proof A calculation shows that $M\rho(S, n) = \rho(S', n)M$. In this calculation the fundamental relation is used. For example Q commutes with B . So to prove that

$$B^i A^2 + B^{i+1}C = QB^i + (1-Q)B^i A$$

we need to show that

$$A^2 + BC = Q + (1-Q)A$$

which follows from the fundamental relation. □

However, if we extend the representation to the **virtual braid group**, defined below, then we get a representation which is **not** equivalent to the Burau.

The virtual braid group, VB_n [KK] , has generators σ_i , $i = 1, \dots, n-1$ and braid group relations

$$\begin{aligned} i) \quad & \sigma_i \sigma_j = \sigma_j \sigma_i, \quad |i-j| > 1 \\ & \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{aligned}$$

In addition there are generators τ_i , $i = 1, \dots, n-1$ and permutation group relations

$$\begin{aligned} ii) \quad & \tau_i^2 = 1 \\ & \tau_i \tau_j = \tau_j \tau_i, \quad |i-j| > 1 \\ & \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \end{aligned}$$

together with mixed relations

$$\begin{aligned} iii) \quad & \sigma_i \tau_j = \tau_j \sigma_i, \quad |i-j| > 1 \\ & \sigma_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \sigma_{i+1} \end{aligned}$$

We can extend the representation $\rho(S, n)$ by sending the generator τ_i to $(id)^{i-1} \times T \times (id)^{n-i-1}$ where $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, that is, Burau with unit variable.

Consider now the element $\beta = \sigma_2 \sigma_1 \tau_2 \sigma_1^{-1} \sigma_2^{-1} \tau_1$ in VB_3 . If $S = \begin{pmatrix} 0 & B \\ C & 1 - BC \end{pmatrix}$, the Alexander switch, then

$$\rho(\beta) = \begin{pmatrix} 1 & 0 & (C^{-1} - B)(B - 1) \\ 0 & 1 & (C^{-1} - B)(1 - B) \\ 0 & 0 & 1 \end{pmatrix}$$

So if $B \neq 1$ this representation can not be equivalent to the Burau representation which has $B = 1$.

We can now ask the following question: *Let $K \subset B_n$ be the kernel of the Burau representation. Let \overline{K} denote the normal closure of K in VB_n . If β is a virtual braid and $\rho(\beta) = 1$ for all switches S , is it true that β lies in \overline{K} ?*

3 Quaternion Algebras

It is clear from the previous section that it is important to find solutions to the fundamental equation. The main result in the next section is a sufficient condition for two generalised quaternions to satisfy the fundamental equation. Except for 2×2 matrices, this condition is also necessary.

In this section we describe the necessary algebra. The results which are already in the literature are mainly presented without proof. For more details see [L] or [MR].

Let F be a field of characteristic not equal to 2. Pick two non-zero elements λ, μ in F . Let $(\frac{\lambda, \mu}{F})$ denote the algebra of dimension 4 over F with basis $\{1, i, j, k\}$ and relations $i^2 = \lambda, j^2 = \mu, ij = -ji = k$. The multiplication table is given by

$$\begin{matrix} & i & j & k \\ \begin{matrix} i \\ j \\ k \end{matrix} & \begin{pmatrix} \lambda & k & \lambda j \\ -k & \mu & -\mu i \\ -\lambda j & \mu i & -\lambda \mu \end{pmatrix} \end{matrix}$$

Throughout the paper a general quaternion algebra will be denoted by \mathcal{Q} . Elements of \mathcal{Q} are called (generalized) quaternions. The field F is called the **underlying** field and the elements λ, μ the **parameters** of the algebra. We will denote quaternions by capital roman letters such as A, B, \dots and (if pure) by bold face lower case, $\mathbf{a}, \mathbf{b}, \dots$. Field elements, (scalars) will be denoted by lower case roman letters such as a, b, \dots and lower case greek letters such as α, β, \dots .

The classical quaternions are $(\frac{-1, -1}{\mathbb{R}})$. The algebra of 2×2 matrices with entries in F is $M_2(F) = (\frac{-1, 1}{F})$.

3.1 Conjugation, Norm and Trace

Let $A = a_0 + a_1i + a_2j + a_3k$ be a quaternion where $a_0, a_1, a_2, a_3 \in F$. The coordinate a_0 is called the **scalar** part of A and the 3-vector $\mathbf{a} = a_1i + a_2j + a_3k$ is called the **pure** part of A . Evidently $A = a_0 + \mathbf{a}$ is the sum of its scalar and pure parts and is pure if its scalar part is zero and is a scalar if its pure part is zero.

The **conjugate** of A is $\bar{A} = a_0 - \mathbf{a}$, the **norm** of A is $N(A) = A\bar{A}$ and the **trace** of A is $\text{tr}(A) = A + \bar{A}$.

Conjugation is an anti-isomorphism of order 2. That is it satisfies

$$\overline{A+B} = \bar{A} + \bar{B}, \quad \overline{AB} = \bar{B}\bar{A}, \quad \overline{aA} = a\bar{A}, \quad \overline{\bar{A}} = A.$$

Also $\bar{A} = A$ if and only if A is a scalar and $\bar{A} = -A$ if and only if A is pure.

The norm is a scalar satisfying $N(AB) = N(A)N(B)$. We will denote the set of values of the norm function by \mathcal{N} . It is a multiplicatively closed subset of F and $\mathcal{N}^* = \mathcal{N} - \{0\}$ is a multiplicative subgroup of F^* . An element A has an inverse if and only if $N(A) \neq 0$ in which case $A^{-1} = N(A)^{-1}\bar{A}$.

The trace of a quaternion is twice its scalar part.

3.2 Multiplying Quaternions

Let A, B be two quaternions. There is a bilinear form given by

$$A \cdot B = \frac{1}{2}(A\bar{B} + B\bar{A}) = \frac{1}{2}(\bar{A}B + \bar{B}A) = \frac{1}{2}\text{tr}(A\bar{B}).$$

In terms of coordinates this is

$$A \cdot B = a_0b_0 - \lambda a_1b_1 - \mu a_2b_2 + \lambda\mu a_3b_3.$$

Since λ and μ are non-zero this is a non-degenerate form. The corresponding quadratic form is

$$N(A) = a_0^2 - \lambda a_1^2 - \mu a_2^2 + \lambda\mu a_3^2.$$

Let \mathbf{a}, \mathbf{b} be pure quaternions. Then

$$\mathbf{a}\mathbf{b} = -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}$$

where

$$\mathbf{a} \cdot \mathbf{b} = -\lambda a_1b_1 - \mu a_2b_2 + \lambda\mu a_3b_3$$

is the restriction of the bilinear form to the pure quaternions and $\mathbf{a} \times \mathbf{b}$ is the **cross product** defined symbolically by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} -\mu i & -\lambda j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

The cross product has the usual rules of bilinearity and skew symmetry. The triple cross product expansion

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}$$

is easily verified. The **scalar triple product** is

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \lambda\mu \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

from which all the usual rules (except volume) can be deduced.

3.3 Dependency Criteria

In this subsection we will consider conditions for sets of quaternions to be linearly dependant or otherwise. A non-zero element, A , of \mathcal{Q} is called **isotropic** if $N(A) = 0$ and **anisotropic** otherwise. So only non-zero anisotropic elements have inverses. We note the following theorem.

Theorem 3.2 *The following statements about a quaternion algebra \mathcal{Q} are equivalent.*

1. \mathcal{Q} contains an isotropic element.
2. \mathcal{Q} is the sum of two hyperbolic planes.
3. \mathcal{Q} is not a division algebra.
4. \mathcal{Q} is isomorphic to $M_2(F)$ as an algebra.

Proof See [L] p 58.

We will call a quaternion algebra above **hyperbolic**. Otherwise it is called **anisotropic**. The classic quaternions are anisotropic: 2×2 matrices are hyperbolic.

Lemma 3.3 *A pair of pure quaternions \mathbf{a}, \mathbf{b} is linearly dependant if and only if $\mathbf{a} \times \mathbf{b} = 0$.*

Proof The proof is clear one way using the antisymmetry of the cross product. Conversely suppose $\mathbf{a} \times \mathbf{b} = 0$. Then $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} = 0$. This can be made into a linear dependency by a suitable choice of \mathbf{c} , for example if $\mathbf{a} \cdot \mathbf{c} \neq 0$. \square

As a corollary we have the following

Lemma 3.4 *Two quaternions commute if and only their pure parts are linearly dependant.*

\square

Now we look for conditions for the triple of pure quaternions, $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$, to be linearly dependant. The required condition is given by the following lemma.

Lemma 3.5 *The pure quaternions $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$, are linearly dependant if and only if*

$$N(\mathbf{a})N(\mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})^2.$$

This is equivalent to the equations

$$N(\mathbf{a} \times \mathbf{b}) = -\mu(a_2b_3 - a_3b_2)^2 - \lambda(a_1b_3 - a_3b_1)^2 + (a_1b_2 - a_2b_1)^2 = 0,$$

ie $\mathbf{a} \times \mathbf{b}$ is isotropic or zero.

Proof Three 3-dimensional vectors are linearly dependant if and only if the determinant they form by rows is zero. In the case of pure quaternions this means the scalar triple product is zero

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \lambda \mu \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Replacing \mathbf{c} with $\mathbf{a} \times \mathbf{b}$ and expanding out using the triple cross product formula gives the first equation. Using the expansion formula

$$N(\mathbf{a} \times \mathbf{b}) = N(\mathbf{a})N(\mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2$$

gives the second formula. □

We have the following corollaries.

Lemma 3.6 *If \mathbf{a}, \mathbf{b} are linearly independant pure quaternions and $\mathbf{a} \times \mathbf{b}$ is anisotropic, then the triple $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$, is linearly independant.*

Lemma 3.7 *\mathcal{Q} is isomorphic to $M_2(F)$ as an algebra if and only if there are linearly independant pure quaternions \mathbf{a}, \mathbf{b} such that the triple $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$, is linearly dependant.*

3.4 2×2 matrices

We will interpret all the previous results in terms of 2×2 matrices, $M_2(F) = \left(\frac{-1, 1}{F}\right)$. This is the only quaternion algebra with zero divisors.

The generators of $\left(\frac{-1, 1}{F}\right)$ are, together with the identity, the Pauli matrices

$$i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

By an abuse of notation we will often confuse the scalar matrix $\begin{pmatrix} \nu & 0 \\ 0 & \nu \end{pmatrix}$ with the corresponding field element ν .

A general matrix can be written uniquely as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{1}{2} [(\alpha + \delta) + (\beta - \gamma)i + (\beta + \gamma)j + (\alpha - \delta)k]$$

Conversely

$$A = a_0 + a_1i + a_2j + a_3k = \begin{pmatrix} a_0 + a_3 & a_2 + a_1 \\ a_2 - a_1 & a_0 - a_3 \end{pmatrix}$$

Conjugation is

$$\bar{A} = \text{adj}A = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} a_0 - a_3 & -a_2 - a_1 \\ a_1 - a_2 & a_0 + a_3 \end{pmatrix}$$

and norm is

$$N(A) = \det A = \alpha\delta - \beta\gamma = a_0^2 + a_1^2 - a_2^2 - a_3^2$$

The scalar part of A is $a_0 = \text{tr}A/2 = (\alpha + \delta)/2$ and the pure part is

$$\begin{pmatrix} a_3 & a_2 + a_1 \\ a_2 - a_1 & -a_3 \end{pmatrix} = \begin{pmatrix} (\alpha - \delta)/2 & \beta \\ \gamma & (\delta - \alpha)/2 \end{pmatrix}$$

3.5 Multiplying Matrices

Lemma 3.2 Suppose $A, B \in M_2(F) = \left(\frac{-1, 1}{F}\right)$. Then $AB = BA$.

The statement is deliberately provocative. It says that multiplying A, B as matrices and as quaternions is the same. This can be checked directly. \square

The above lemma allows quick checking of formulæ so if

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} \text{ and } B = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \text{ then } A \cdot B = \frac{1}{2}(\alpha_1\beta_4 - \alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_4\beta_1)$$

If $\mathbf{a} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & -\alpha_1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & -\beta_1 \end{pmatrix}$ are pure then $\mathbf{a} \cdot \mathbf{b} = -\alpha_1\beta_1 - (\alpha_2\beta_3 + \alpha_3\beta_2)/2$ and

$$\mathbf{a} \times \mathbf{b} = \begin{pmatrix} (\alpha_2\beta_3 - \alpha_3\beta_2)/2 & \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 & (\alpha_3\beta_2 - \alpha_2\beta_3)/2 \end{pmatrix}$$

4 Solving the Fundamental Equation

In this section we give a sufficient condition for two generalised quaternions to satisfy the fundamental equation. Except for 2×2 matrices, this condition is also necessary.

Let $A = a_0 + \mathbf{a}$ and $B = b_0 + \mathbf{b}$ be quaternions. We will need the following easily checked lemma.

Lemma 4.3 Conjugation by multiplication is $B^{-1}AB = N(B)^{-1}\bar{B}AB$ where

$$\bar{B}AB = a_0(b_0^2 + N(\mathbf{b})) + (b_0^2 - N(\mathbf{b}))\mathbf{a} + 2(\mathbf{a} \cdot \mathbf{b})\mathbf{b} + 2b_0(\mathbf{a} \times \mathbf{b}).$$

The two commutators are

$$[A, B] = AB - BA = 2\mathbf{a} \times \mathbf{b}$$

and $(A, B) = A^{-1}B^{-1}AB = N(A)^{-1}N(B)^{-1}\overline{A}\overline{B}AB$ where

$$\begin{aligned}\overline{A}\overline{B}AB &= a_0^2b_0^2 + b_0^2N(\mathbf{a}) + a_0^2N(\mathbf{b}) - N(\mathbf{a})N(\mathbf{b}) + 2(\mathbf{a} \cdot \mathbf{b})^2 \\ &\quad - 2(b_0(\mathbf{a} \cdot \mathbf{b}) + a_0N(\mathbf{b}))\mathbf{a} + 2(a_0(\mathbf{a} \cdot \mathbf{b}) + b_0N(\mathbf{a}))\mathbf{b} + 2(a_0b_0 - \mathbf{a} \cdot \mathbf{b})\mathbf{a} \times \mathbf{b}\end{aligned}$$

□

We will call a quaternion, A , **balanced** if $\text{tr}(A) = N(A) \neq 0$. A quaternion $A = a_0 + a_1i + a_2j + a_3k$ in a quaternion algebra with parameters λ, μ is balanced if it lies on the quadric 3-fold

$$(a_0 - 1)^2 - \lambda a_1^2 - \mu a_2^2 + \lambda \mu a_3^2 = 1, \quad a_0 \neq 0.$$

A balanced classical quaternion A lies on the 3-sphere, centre 1 and radius 1. Note that if A is balanced then $N(A - 1) = 1$. A pair of invertible non-commuting quaternions, A, B , will be called **matching** if A is balanced and $A \cdot B = 0$.

Theorem 4.4 *If the quaternion algebra is anisotropic then a necessary and sufficient condition for the non-commuting, invertible quaternions A, B to be solutions of the fundamental equation is that they are a matching pair. Otherwise the condition is only sufficient.*

Proof The proof formally follows [BuF]. In terms of quaternions the equation is

$$\overline{A}\overline{B}AB - B\overline{A}\overline{B}A = N(A)\overline{B}AB - N(A)N(B)A.$$

Using the formulæ and notation developed above, the left hand side is

$$-4a_0N(\mathbf{b})\mathbf{a} + 4a_0(\mathbf{a} \cdot \mathbf{b})\mathbf{b} - 4(\mathbf{a} \cdot \mathbf{b})\mathbf{a} \times \mathbf{b}$$

whereas the right hand side is

$$-2(a_0^2 + N(\mathbf{a}))N(\mathbf{b})\mathbf{a} + 2(a_0^2 + N(\mathbf{a}))(\mathbf{a} \cdot \mathbf{b})\mathbf{b} + 2b_0(a_0^2 + N(\mathbf{a}))\mathbf{a} \times \mathbf{b}$$

Considering half the difference of the two sides we arrive at

$$\mathbf{c} = (\text{tr}(A) - N(A))N(\mathbf{b})\mathbf{a} + (N(A) - \text{tr}(A))(\mathbf{a} \cdot \mathbf{b})\mathbf{b} + (b_0(N(A) - \text{tr}(A)) + 2A \cdot B)\mathbf{a} \times \mathbf{b}$$

So the two sides are equal if $\mathbf{c} = 0$.

Note that we are assuming A, B do not commute. So $\mathbf{a} \times \mathbf{b} \neq 0$. If A, B is a matching pair then $\mathbf{c} = 0$. Conversely if $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ are linearly independent then their coefficients will be zero and this implies that A, B is a matching pair. The bilinear form is definite unless the algebra is $M_2(F)$ and so, except for this case, $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ will be linearly independent. □

4.1 The Non-definite Case

The condition for linear dependency can be satisfied for the case when the quaternions are 2×2 matrices, ie $\lambda = -1, \mu = 1$. Let us try

$$a_2b_3 - a_3b_2 = 1, \quad a_1b_3 - a_3b_1 = 1, \quad a_1b_2 - a_2b_1 = 0$$

One solution is

$$\mathbf{a} = i + tj - k, \mathbf{b} = j \text{ and } \mathbf{a} \times \mathbf{b} = k - i$$

where t is an arbitrary field element. So

$$\mathbf{a} - t\mathbf{b} + \mathbf{a} \times \mathbf{b} = 0$$

We will return to this case in the next paper.

5 The general matching pair

In this section we will use the results of the previous section to describe the most general matching pair of matrices. That is A, B are 2×2 matrices with entries in some field and satisfying

1. $\text{tr}(A) = \det(A) (\neq 0)$
2. $A\text{adj}(B) + B\text{adj}(A) = 0$

Note that condition 2. can also be written $\text{tr}(AB^{-1}) = 0$

Since B can be multiplied by any non-zero scalar we may assume temporarily that

3. $\det(B) = 1.$

We can conjugate the matrices A, B to simplify matters. Consider the following two cases:

Case 1, A has two distinct eigenvalues and is diagonal.

$$A = \begin{pmatrix} a & 0 \\ 0 & a/(a-1) \end{pmatrix} \text{ and } B = \begin{pmatrix} b & c \\ (b^2 + a - 1)/c(1-a) & b/(1-a) \end{pmatrix}$$

where a, b, c are general and $c, 1 - a$ must be invertible (ie non zero).

Inverses are given by

$$A^{-1} = \frac{1}{a} \begin{pmatrix} 1 & 0 \\ 0 & a-1 \end{pmatrix}$$

so a must also be invertible and

$$B^{-1} = \text{adj}(B) = \begin{pmatrix} b/(1-a) & -c \\ -(b^2 + a - 1)/c(1-a) & b \end{pmatrix}.$$

The 4×4 matrix S is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where

$$C = A^{-1}B^{-1}A - A^{-1}B^{-1}A^2 = \begin{pmatrix} b & c/(1-a)^2 \\ (1-a)(b^2 + a - 1)/c & b/(1-a) \end{pmatrix}$$

and

$$D = 1 - A^{-1}B^{-1}AB = \begin{pmatrix} (2 - 3a + ab^2 + a^2 - 2b^2)/(1 - a)^2 & (a - 2)bc/(1 - a)^2 \\ (a - 2)b(b^2 + a - 1)/c(1 - a) & (2 - 3a + ab^2 + a^2 - 2b^2)/(1 - a) \end{pmatrix}$$

Call this switch \mathbf{E}_2

Case 2 A has one eigenvalue and is lower triangular

$$A = \begin{pmatrix} 2 & 0 \\ x & 2 \end{pmatrix}.$$

Then B has the form,

$$B = \begin{pmatrix} y & z \\ (xyz - 2y^2 - 2)/2z & (xz - 2y)/2 \end{pmatrix}$$

where x, y, z are general and $2, z$ must be invertible.

Inverses are given by

$$A^{-1} = \frac{1}{4} \begin{pmatrix} 2 & 0 \\ -x & 2 \end{pmatrix}.$$

and

$$B^{-1} = \text{adj}(B) = \begin{pmatrix} (xz - 2y)/2 & -z \\ (-xyz + 2y^2 + 2)/2z & y \end{pmatrix}$$

The 4×4 matrix S is $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ where

$$C = A^{-1}B^{-1}A - A^{-1}B^{-1}A^2 = \begin{pmatrix} y + xz & z \\ -(x^2z^2 + 3xyz + 2y^2 + 2)/2z & -xz/2 - y \end{pmatrix}$$

and

$$D = 1 - A^{-1}B^{-1}AB = \begin{pmatrix} xyz/2 & xz^2/2 \\ -x(2y^2 + xyz - 2)/4 & -xz(xz + 2y)/4 \end{pmatrix}$$

Call this switch \mathbf{E}_1

5.1 The variable t

Let us now multiply B by the scalar t , ie the switch is augmented by t . This means we have two possible switches: one dependant on four variables a, b, c, t (E_2) and one dependant on four variables x, y, z, t (E_1).

6 Determinants over Quaternion Algebras

In order to define workable invariants we consider in this section a determinantal function on the matrices in $M_n(\mathcal{Q})$. That is $n \times n$ matrices with entries in a quaternion algebra \mathcal{Q} . For background reading see [As]. In fact the invariants defined later can also be defined for any solutions of the fundamental equation over a ring with a determinant function satisfying the rules listed below.

If R is a commutative ring let $\det : M_n(R) \rightarrow R$ denote the usual determinant. The classic quaternions, \mathcal{H} , may be embedded as a subalgebra of $M_2(\mathbb{C})$ and determinants taken in the usual way. Our aim is to generalize this.

Suppose \mathcal{Q} has underlying field F and parameters λ, μ . Let \overline{F} denote the algebraic closure of F . Embed i, j, k in $M_2(\overline{F})$ by

$$i = \begin{pmatrix} 0 & \sqrt{-\lambda} \\ -\sqrt{-\lambda} & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & \sqrt{\mu} \\ \sqrt{\mu} & 0 \end{pmatrix}, \quad k = \begin{pmatrix} \sqrt{-\lambda\mu} & 0 \\ 0 & -\sqrt{-\lambda\mu} \end{pmatrix}.$$

Define $d : M_n(\mathcal{Q}) \rightarrow \overline{F}$ as the composition of the embedding $M_n(\mathcal{Q}) \subset M_{2n}(\overline{F})$ with \det .

Alternatively the determinant function may be calculated by induction on the size of the matrices. The value $d(A) = N(A)$ starts the induction. Consider a matrix in $M_n(\mathcal{Q})$. This may be reduced to diagonal form, by multiplying on the left and the right by elementary matrices having unit determinant, (see below). Suppose this matrix has diagonal elements d_1, \dots, d_n . Define the determinant as $d = N(d_1) \cdots N(d_n)$. So the determinant function takes values in \mathcal{N} . For $M_2(F)$ this subset is the whole of F : for classic quaternions it is the non-negative reals.

The determinant function satisfies the rules

0. $d(M) = 0$ if and only if M is singular, moreover $d(MN) = d(M)d(N)$. It follows that $d(1) = 1$.
1. d is unaltered by a permutation of the rows (columns).
2. If a row (column) is multiplied on the left (right) by a unit then d is multiplied by d of that unit.
3. $d(M)$ is unaltered by adding a left multiple of a row to another row or a right multiple of a column to another column.
4. $d \begin{pmatrix} x & \mathbf{u} \\ \mathbf{0} & M \end{pmatrix} = N(x)d(M)$ where \mathbf{u} is any row vector and $\mathbf{0}$ is a zero column vector.
- 4'. $d \begin{pmatrix} x & \mathbf{0} \\ \mathbf{v} & M \end{pmatrix} = N(x)d(M)$ where \mathbf{v} is any column vector and $\mathbf{0}$ is a zero row vector.
5. $d(M^*) = d(M)$ where $M^* = \overline{M^T}$ denotes the Hermitian conjugate.
6. if the entries in M all commute then $d(M) = \det^2(M)$.

An **elementary** matrix of type 1 is a permutation matrix. An elementary matrix of type 2 is the identity matrix with one diagonal entry replaced by a unit and an elementary matrix of type 3 is a square matrix with zero entries except for 1's down the diagonal and one other entry off diagonal.

The properties i . above for $i = 1, 2, 3$ follow from multiplying M on the right or left by an elementary matrix of type i .

The matrix S can be written as a product of elementary matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - A^{-1} \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}.$$

Note that $1 - A^{-1}$ is invertible.

Hence $d(S) = d(A)d(1 - A^{-1}) = d(A - 1) = 1$. Therefore the representation of VB_n , induced by such an S , is into $SL(F, 2n)$.

7 Virtual Knots and Links

Recall that classical knot theory can be described in terms of knot and link diagrams. A **diagram** is a 4-regular plane graph (with extra structure at its nodes representing the crossings in the link) on a plane and implicitly on a two-dimensional sphere S^2 . We say that two such diagrams are **equivalent** if there is a sequence of moves of the types indicated in part (A) of Figure 1 (The Reidemeister Moves) taking one diagram to the other. These moves are performed locally on the 4-regular plane graph (with extra structure) that constitutes the link diagram.

Virtual knot theory is an extension of classical knot theory, see [K]. In this extension one adds a **virtual crossing** (See Figure 1) that is neither an over-crossing nor an under-crossing. We shall refer to the usual diagrammatic crossings, that is those without circles, as **real** crossings to distinguish them from the virtual crossings. A virtual crossing is represented by two crossing arcs with a small circle placed around the crossing point. The arcs of the graph joining adjacent classical crossings are called the **semi-arcs** of the diagram.

In addition to their application as a geometric realization of the combinatorics of a Gauss code, virtual links have physical, topological and homological applications. In particular, virtual links may be taken to represent a particle in space and time which disappears and reappears. A virtual link may be represented, up to stabilisation, by a link diagram on an orientable surface, [Ku]. If the surface has minimal genus then this representation is unique. Finally an element of the second homology of a rack space can be represented by a labelled virtual link, see [FRS1] [FRS2]. Since the rack spaces form classifying spaces for classical links the study of virtual links may give information about classical knots and links.

The allowed moves on virtual diagrams are a generalization of the Reidemeister moves for classical knot and link diagrams. We show the classical Reidemeister moves as part (A) of Figure 1. These classical moves are part of virtual equivalence where no changes are made to the virtual crossings. Taken by themselves, the virtual crossings behave as diagrammatic permutations. Specifically, we have the flat Reidemeister moves (B) for virtual crossings as shown in Figure 1. In Figure 1 we also illustrate a basic move (C) that interrelates real and virtual crossings. In this move an arc going through a consecutive sequence of two virtual crossings can be moved across a single real crossing. In fact, it is consequence of moves (B) and (C) for virtual crossings that an arc going through any consecutive sequence of virtual

crossings can be moved anywhere in the diagram keeping the endpoints fixed and writing the places where the moved arc now crosses the diagram as new virtual crossings. This is shown schematically in Figure 2. We call the move in Figure 2 the **detour**, and note that the detour move is equivalent to having all the moves of type (B) and (C) of Figure 1. This extended set of moves (Reidemeister moves plus the detour move or the equivalent moves (B) and (C)) constitutes the set of moves for diagrams of virtual knots and links.

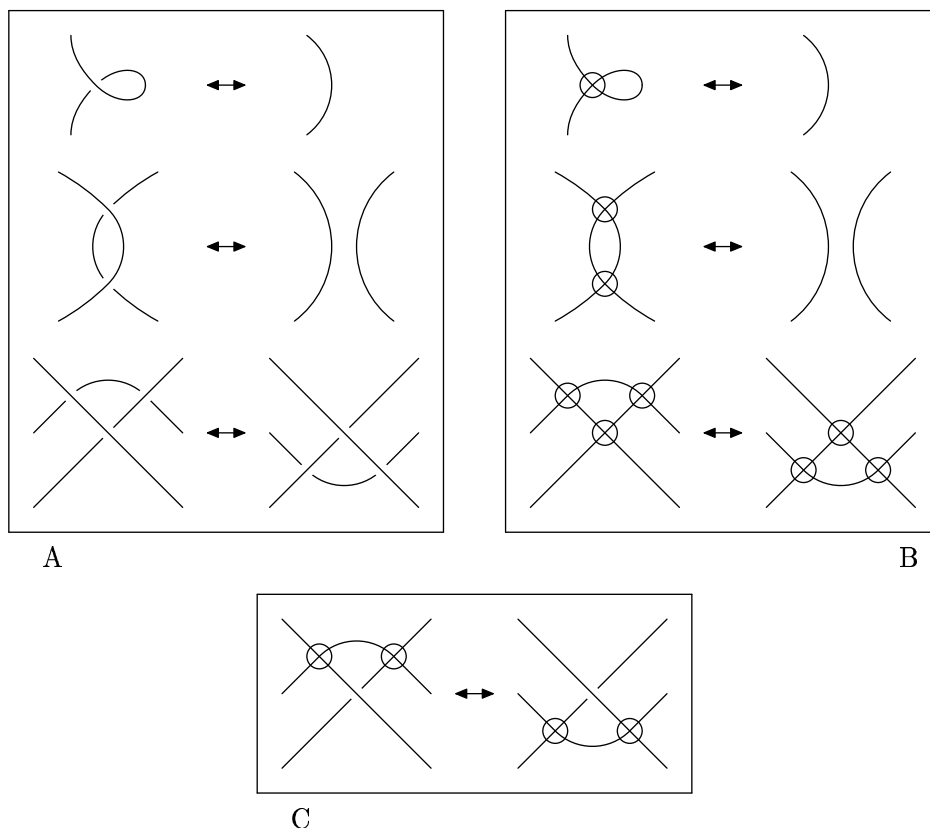


fig1: Generalized Reidemeister Moves for Virtual Knots

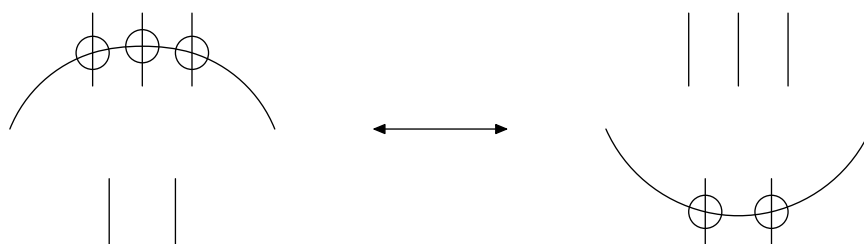


fig2: Illustration of the Detour Move

The topological interpretation for this virtual theory in terms of embeddings of links in thickened surfaces is a useful idea. See [KK], [Ku]. Regard each virtual crossing as a shorthand for a detour of one of the arcs in the crossing through a 1-handle that has been attached to the 2-sphere of the original diagram. The two choices for the 1-handle detour are homeomorphic to each other (as abstract surfaces with boundary a circle) since there is no a priori difference

between the meridian and the longitude of a torus. By interpreting each virtual crossing in this way, we obtain an embedding of a collection of circles into a thickened surface $S_g \times \mathbb{R}$ where g is the number of virtual crossings in the original diagram L , S_g is a compact oriented surface of genus g and \mathbb{R} denotes the real line. Thus to each virtual diagram L we obtain an embedded disjoint union of circles in $S_{g(L)} \times \mathbb{R}$ where $g(L)$ is the number of virtual crossings of L . We say that two such surface embeddings are *stably equivalent* if one can be obtained from another by isotopy in the thickened surfaces, homeomorphisms of the surfaces and the addition or subtraction of empty handles. Then we have the

Theorem 7.2 *Two virtual link diagrams are equivalent if and only if their correspondent surface embeddings are stably equivalent, [KK] , [Ku] .*

The surface embedding interpretation of virtuals is useful since it converts their equivalence to a topological question. The diagrammatic version of virtuals embodies the stabilization in the detour moves. We shall rely on the diagrammatic approach here.

8 The invariant knot modules

In this section, we shall begin to show how the previous algebra can give rise to virtual knot invariants. Given an associative ring R , a 2×2 matrix S with entries in R and a virtual link diagram \mathcal{D} , we define a presentation of an R -module which depends only on the link class of the diagram provided S is a switch. This construction also works for classical knots and links but is only the Alexander module in disguise. The generators are the semi-arcs of \mathcal{D} , that is the portion of the diagram bounded by two adjacent classical crossings. There are 2 relations for each classical crossing.

Suppose the diagram \mathcal{D} has n classical crossings. Then there are $2n$ semi-arcs labelled a, b, \dots . These will be the generators of the module. Let the edges of a positive real crossing in a diagram be arranged diagonally and called geographically *NW*, *SW*, *NE* and *SE*. Assume that initially the crossing is oriented and the edges oriented towards the crossing from left to right ie west to east. The **input** edges, oriented towards the crossing, are in the west and the edges oriented away from the crossing, the **output** edges, are in the east. Let a and b be the labels of the input edges with a labelling SW and b labelling NW. For a positive crossing, a will be the label of the undercrossing input and b the label of the overcrossing input. Suppose now that

$$S(a, b)^T = (c, d)^T \text{ where } S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then we label the undercrossing output NE by d and we label the overcrossing output SE by c .

For a negative crossing the direction of labelling is reversed. So a labels SE, b labels NE, c labels SW and d labels NW.

Finally for a virtual crossing the labellings carry across the strings. This corresponds to the twist function $T(a, b) = (b, a)$.

The following figure shows the labelling for the three kind of crossings and the corresponding

relations for the 2 classical crossings.

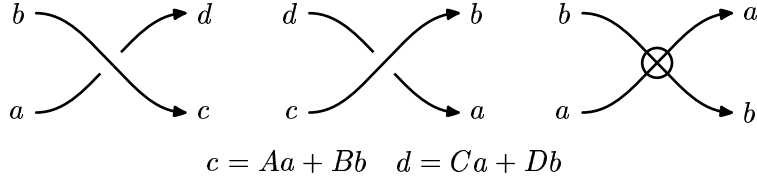


fig 3: Labelling the various crossings

The diagram therefore gives rise to a presentation of an R -module with $2n$ generators and $2n$ relations. Note that in all cases B, C are invertible since the identity switch is uninteresting.

Theorem 8.3 *The module defined above for any diagram D is invariant under the Reidemeister moves, and hence is a knot invariant, if S is a switch.*

Proof The proof of the above can be found in the papers [FJK, BF, BuF] , For the convenience of the reader we show how the module is invariant under the three Reidemeister moves.

Referring back to the picture of the two relations defined by a crossing, it is convenient to think of the action from left to right on a positive crossing as being the action of S and the action from right to left as being S^{-1} .

Consider the action from top to bottom as being S_+^- and the action from bottom to top as being S_+^+ . By solving the equations of the labellings we see that these matrices are

$$S_-^+ = \begin{pmatrix} DB^{-1} & C - DB^{-1}A \\ B^{-1} & -B^{-1}A \end{pmatrix} \quad S_+^- = \begin{pmatrix} -C^{-1}D & C^{-1} \\ B - AC^{-1}D & AC^{-1} \end{pmatrix}$$

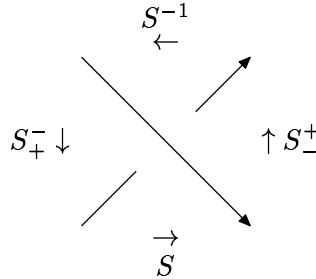


fig 4: Direction of the S operators

We call S_-^+ and S_+^- the **sideways** matrices. They are invertible since S is. Also $(S^{-1})_-^+ = (S_+^-)^{-1}$ and $(S^{-1})_+^- = (S_-^+)^{-1}$ and

$$S_-^+(a, a) = (\lambda a, \lambda a) \text{ and } S_+^-(a, a) = (\lambda^{-1}a, \lambda^{-1}a)$$

where $\lambda = B^{-1}(1 - A) = (1 - D)^{-1}C$. So the sideways matrices preserve the diagonal. This has the curious consequence that a linear switch which is a birack is also a biquandle in the sense of [FJK] .

For a negative crossing the actions are equal but with opposite orientation.

Assume for simplicity that we are dealing with a knot. The link case is similar and details can safely be left to the reader. We have a right R -module with a finite (square) presentation.

Following the orientation of the knot, label the semi-arcs with R -variables x_1, x_2, \dots, x_{2n} . By an R -variable we mean a symbol standing in for any element of R .

At each crossing there is a relation of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_i \\ x_j \end{pmatrix} = \begin{pmatrix} x_{j+1} \\ x_{i+1} \end{pmatrix} \text{ or } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_i \\ x_j \end{pmatrix} = \begin{pmatrix} x_{j-1} \\ x_{i-1} \end{pmatrix}$$

depending on whether the crossing is positive or negative. As is the usual custom, indices are taken modulo $2n$.

The relations can now be written in matrix form as $M\mathbf{x} = \mathbf{0}$ where M is a $2n \times 2n$ matrix and $\mathbf{x} = (x_1, x_2, \dots, x_{2n})^T$. The non-zero entries in each row of the matrix are $A, B, -1$ or $C, D, -1$.

Let $\mathcal{M} = \mathcal{M}(S, D)$ be the module defined by these relations. We now show that the modules defined by diagrams representing the same virtual link are isomorphic. We do this by showing that a single Reidemeister move defines an isomorphism. The proof has the same structure as the proof, say, that the Alexander module of a classical link is an invariant as in [Alex] but we give the details because of the care needed due to non-commutativity.

Any module defined by a presentation of the form $M\mathbf{x} = \mathbf{0}$ is invariant under the following moves and their inverses applied to the matrix M .

1. permutations of rows and columns,
2. multiplying any row on the left or any column on the right by a unit,
3. adding a left multiple of a row to another row or a right multiple of a column to another column,
4. changing M to $\begin{pmatrix} x & \mathbf{u} \\ \mathbf{0} & M \end{pmatrix}$ where x is a unit, \mathbf{u} is any row vector and $\mathbf{0}$ is a zero column vector,
5. repeating a row.

The operations i . above for $i = 1, 2, 3$ are equivalent to multiplying M on the right or left by an elementary matrix of type i .

Recall that the matrix S can be written as a product of elementary matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 - A^{-1} \end{pmatrix} \begin{pmatrix} 1 & A^{-1}B \\ 0 & 1 \end{pmatrix}.$$

Now consider the module \mathcal{M} defined above. Clearly the presentation is unaltered by any of the basic moves which involve the virtual crossing. So we look to see the changes induced by the classical Reidemeister moves and check that the presentation matrix M is only changed by the above 5 moves. Assume

$$M = \begin{pmatrix} m_{11} & \cdots & m_{12n-1} & m_{12n} \\ m_{21} & \cdots & m_{22n-1} & m_{22n} \\ \vdots & \ddots & \vdots & \vdots \\ m_{2n1} & \cdots & m_{2n2n-1} & m_{2n2n} \end{pmatrix}$$

Firstly, consider a Reidemeister move of the first kind.

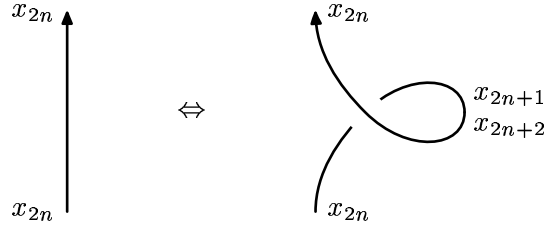


fig 5: Reidemeister I

This introduces (or deletes) two new equal generators $x_{2n+1} = x_{2n+2}$. Here we have conveniently reordered the generators so that the two new ones appear last.

Because S_+^- and S_-^+ preserve the diagonal, (the biquandle condition, see [FJK]) the output (x_{2n+3}) is the same as the input (x_{2n}). The generator x_{2n+1} is equal to $\lambda^{-1}x_{2n}$ where $\lambda = B^{-1}(1 - A)$.

So up to reordering of the columns the relation matrix is changed by

$$M \Leftrightarrow \begin{pmatrix} m_{11} & \cdots & m_{12n-1} & m_{12n} & 0 & 0 \\ m_{21} & \cdots & m_{22n-1} & m_{22n} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ m_{2n1} & \cdots & m_{2n2n-1} & m_{2n2n} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 & -1 \\ 0 & \cdots & 0 & 1 & -\lambda & 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} m_{11} & \cdots & m_{12n-1} & m_{12n} & 0 & 0 \\ m_{21} & \cdots & m_{22n-1} & m_{22n} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ m_{2n1} & \cdots & m_{2n2n-1} & m_{2n2n} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & -1 \\ 0 & \cdots & 0 & 1 & \lambda & 0 \end{pmatrix}$$

Since λ is a unit this does not alter the module.

There are other possible inversions and mirror images of the above which can be dealt with in a similar fashion.

Secondly, consider a Reidemeister move of the second kind.

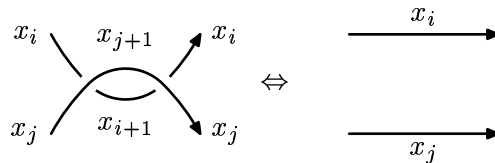


fig 6: Reidemeister II

Again the outputs are unchanged from the inputs x_i, x_j because of the relation $S^{-1}S = 1$.

Two new generators x_{i+1} and x_{j+1} are introduced (or deleted). They are related by the equations

$$x_j = Ax_{i+1} + Bx_{j+1} \text{ and } x_i = Cx_{i+1} + Dx_{j+1}.$$

After reordering of the rows and columns this has the following effect on the relation matrix.

$$M \Leftrightarrow \begin{pmatrix} m_{11} & \dots & m_{12n-1} & m_{12n} & 0 & 0 \\ m_{21} & \dots & m_{22n-1} & m_{22n} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ m_{2n1} & \dots & m_{2n2n-1} & m_{2n2n} & 0 & 0 \\ 0 & \dots & 0 & -1 & A & B \\ 0 & \dots & -1 & 0 & C & D \end{pmatrix}.$$

Since S is a product of elementary matrices this does not alter the module.

The other possible inversions and mirror images of the above can be dealt with in a similar fashion but it is worth looking at the case where the two arcs run in opposite directions. The right outputs are unchanged from the left inputs by the relation $S_-^+(S_-^{-1})_-^+ = 1$.

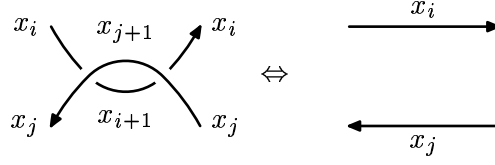


fig 7: Reidemeister II (opposite directions)

The changes to the relation matrix are given by

$$M \Leftrightarrow \begin{pmatrix} m_{11} & \dots & m_{12n-1} & m_{12n} & 0 & 0 \\ m_{21} & \dots & m_{22n-1} & m_{22n} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ m_{2n1} & \dots & m_{2n2n-1} & m_{2n2n} & 0 & 0 \\ 0 & \dots & -1 & A & B & 0 \\ 0 & \dots & 0 & C & D & -1 \end{pmatrix} \Leftrightarrow \begin{pmatrix} m_{11} & \dots & m_{12n-1} & m_{12n} & 0 & 0 \\ m_{21} & \dots & m_{22n-1} & m_{22n} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ m_{2n1} & \dots & m_{2n2n-1} & m_{2n2n} & 0 & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & -1 \end{pmatrix}$$

using the fact that B is a unit. The change doesn't alter the module.

Finally, consider a Reidemeister move of the third kind.

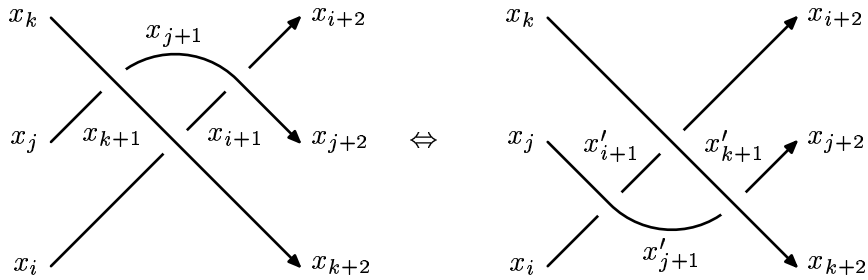


fig 8: Reidemeister III

The outputs $x_{i+2}, x_{j+2}, x_{k+2}$ are unaltered by the Reidemeister move because of the Yang-Baxter equations. The inner generators $x_{i+1}, x_{j+1}, x_{k+1}$ are related to the inputs x_i, x_j, x_k by the following matrix

$$\begin{pmatrix} C & DA & DB \\ 0 & C & D \\ 0 & A & B \end{pmatrix}$$

and the inner generators $x'_{i+1}, x'_{j+1}, x'_{k+1}$ are related to x_i, x_j, x_k by the following matrix

$$\begin{pmatrix} C & D & 0 \\ A & B & 0 \\ AC & AD & B \end{pmatrix}.$$

Both are the product of elementary matrices and the proof is finished. \square

9 Determinant Invariants

9.1 The Determinant Δ_0

Given a module with a square presentation the obvious invariant of the module is the determinant, if it can be defined. This will be the case if the ring is represented by matrices with commuting entries, for example the ring of generalised quaternions. In this case if d denotes the determinant and $M\mathbf{x} = 0$ is the presentation let $\Delta_0 = d(M)$. Since the module depends on the switch S we illustrate this dependency by $\Delta_0 = \Delta_0(S)$.

A close look at how the presentation of the module changes under the Reidemeister moves shows that Δ_0 is invariant up to multiplication by $d(B)$ or $d(C)$. Typically $d(B)$ is denoted by the variable t and $d(C)$ is t^{-1} . If we take the switch to be E_1 (E_2) defined in section 5 then Δ_0 is a polynomial p_1 (p_2) in the four variables x, y, z, t (a, b, c, t). We can normalise these polynomial so that as a polynomial in t it has a non-zero constant term and only positive powers of t .

Let us illustrate the previous discussion by calculating invariants for the **virtual trefoil** as shown in the figure.

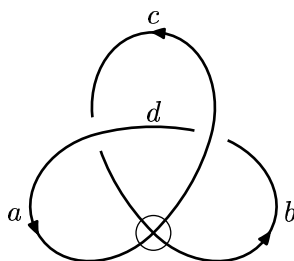


fig 9: The virtual Trefoil

If we label as indicated then the module has a presentation with 4 generators a, b, c, d and relations $c = Ab + Ba$, $a = Ac + Bd$, $b = Cc + Dd$, $d = Cb + Da$. Restricting to the E_1 case gives the polynomial p_1 equal to

$$\frac{64 + 4x^3tz^3 + 4x^3t^3z^3 + 128t^2 - 64xtz - x^4t^2z^4 - 64xt^3z + 8x^2t^2z^2 - 4x^2z^2 + 64t^4 - 4x^2t^4z^2}{16}$$

16

21

For the E_2 case we get the polynomial p_2 equal to

$$\frac{-4 + 16a - 40tba^3 - 4t^4a^4 + 16t^4a^3 - 24t^4a^2 + 16t^4a - 36tb^3a^2 + 72a^2tb + 4b^2 + 13a^2t^4b^2 + 40tb^3a + 4t^4b^2 - 4t^4 - 24a^2 - 12b^2a + 13a^2b^2 + 16a^3 - 12t^4b^2a + 16tb - 6t^4b^2a^3 + t^4b^2a^4 - 56atb - 16tb^3 + b^2a^4 - 6b^2a^3 + 8ba^4t - 2b^3a^4t + 14b^3a^3t - 4a^4 - 8t^2a^4 + 32t^2a^3 - 48t^2a^2 + 32t^2a - 8t^2 + a^4t^2b^4 - 2a^4t^2b^2 - 8a^3t^2b^4 - 8t^2b^2 + 16t^2b^4 + 16t^3b - 16t^3b^3 - 26a^2t^2b^2 + 12a^3t^2b^2 + 24a^2t^2b^4 + 8a^4t^3b - 40a^3t^3b + 72a^2t^3b + 24at^2b^2 - 32at^2b^4 - 56at^3b - 2t^3b^3a^4 + 14t^3b^3a^3 - 36t^3b^3a^2 + 40t^3b^3a}{t^2(a-1)^4}$$

Note that the fundamental quandle (and hence group) as defined by the Wirtinger presentation is trivial.

The following virtual knot is interesting in having a trivial Jones-polynomial as well as a trivial fundamental rack.

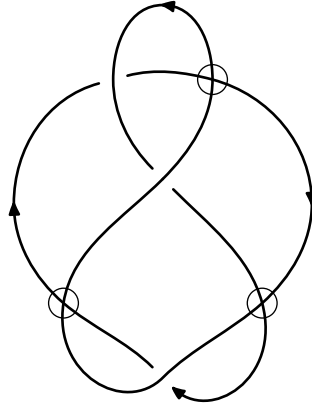


fig 10: A knot with trivial Jones-polynomial

In this case if S is the Alexander switch then

$$\Delta_0 = (B - 1)(C^2(B + 1) - C(B + 1)(B^{-1} + 1) - B).$$

9.2 The Determinant Δ_1

For many knots and links, including the classical, the determinant Δ_0 is zero.

For example, as we have seen earlier any switch S with entries in the ring R defines a representation of the virtual braid group VB_n into the group of invertible $n \times n$ matrices with entries in R by sending the standard generator σ_i to $S_i = (id)^{i-1} \times S \times (id)^{n-i-1}$ and the generator τ_i to $T_i = (id)^{i-1} \times T \times (id)^{n-i-1}$. This representation is denoted by $\rho = \rho(S, n)$. For classical braids this representation is equivalent to the Burau representation and so we would expect the closure of a classical braid to have Δ_0 zero. We now confirm this by looking at the fixed points of S_i both on the left and right.

Lemma 9.2 *Let $P = A^{-1}B^{-1}A$ and $Q = B^{-1}(1 - A)$. Then*

$$(P^{n-1}, \dots, P, 1)S_i = (P^{n-1}, \dots, P, 1) \quad (*)$$

and

$$S_i(1, Q, \dots, Q^{n-1})^T = (1, Q, \dots, Q^{n-1})^T \quad (**).$$

Proof We need only check that

$$(P, 1)S = (P, 1) \text{ and } S(1, Q)^T = (1, Q)^T.$$

□

Therefore the following lemma gives a necessary condition for the knot or link to be classical.

Theorem 9.3 For all classical knots and links $\Delta_0 = 0$.

Proof Since Δ_0 is an invariant of the module we can assume that the diagram from which it is defined is the closure of a braid. However from * (or **) there is a linear relationship amongst the rows (columns), so $\Delta_0 = 0$. □

The Kishino knots K_1, K_2 and K_3 are illustrated below.

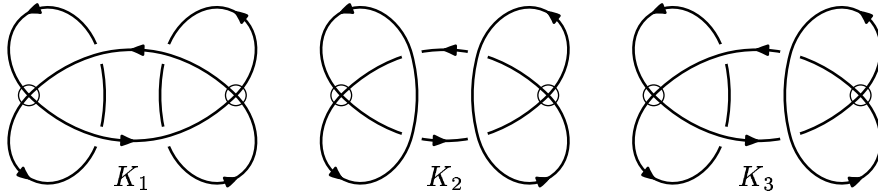


fig 11: The Kishino Knots

All are ways of forming the connected sum of two unknots. K_1 and K_2 are mirror images and K_3 is amphichæral. Both have trivial racks and Jones polynomial. The invariant Δ_0 is zero in all three cases.

It is clear that we need an invariant for these cases. Let M be the $2n \times 2n$ presentation matrix. Let $M_1, M_2, \dots, M_{(2n)^2}$ be the submatrices obtained by deleting a row and a column. Let $d_1, d_2, \dots, d_{(2n)^2}$ be the determinants. These all lie in a ring of polynomials with coefficients in a field. Therefore the determinants have an hcf, call it Δ_1 , which is well defined up to multiplication by a unit. Now look at what happens to this construction under a Reidemeister I move. The hcf, Δ_1 , is multiplied or divided by d of a unit.

Returning to the Kishino knots, a calculation with the Alexander switch shows that for K_1 , Δ_1 is $1 + B - CB$ and for K_2 , Δ_1 is $1 + C - CB$. Since these are neither units nor associates in the ring, K_1, K_2 are non-trivial and non amphichæral.

On the other hand for K_3 , Δ_1 is 1. We will show shortly that K_3 is non-trivial by using the Budapest switch augmented by t . Then $\Delta_1 = 2 + 5t^2 + 2t^4$ see [BuF].

For a classical knot or link the invariant Δ_1 is not just the Alexander polynomial in disguised form but is independant of the deleted rows or columns chosen, up to multiplication by a power of t .

Theorem 9.4 *Let \mathcal{D} be a diagram of a classical knot or link and let M be the presentation matrix associated with \mathcal{D} . Consider $d(M_{ij})$ where M_{ij} is obtained from M by deleting the i th row and the j th column. Then $d(M_{ij})$ is independent of i, j up to multiplication by a power of t and is equal to Δ_1 .*

Proof Assume initially that \mathcal{D} is the closure of a braid.

Write

$$M = (C_1 \quad C_2 \quad \dots \quad C_{2n})$$

in terms of its columns and let

$$M_{ij} = (C_1^0 \quad C_2^0 \quad \dots \quad C_{2n}^0)$$

where each column has its i th component removed and C_j^0 does not appear in the list. From **,

$$C_j^0 = -C_1^0 Q^{(1-j)} - \dots - C_{2n}^0 Q^{2n-j}$$

and C_j^0 does not appear on the right hand side of the equation.

So by column operations which do not change the value of the determinant we can change any column to C_j^0 . Now note that the value of the determinant is unchanged by interchanging two columns. A similar argument works for the rows.

A general diagram is obtained from \mathcal{D} by a sequence of Reidemeister moves. A glance at the change of M under the Reidemeister moves shows that Δ_1 is invariant up to multiplication by a power of t . \square

Let us now return to K_3 . This is the closure of the braid

$$\tau_2(\sigma_1\sigma_2\sigma_1)\tau_2(\sigma_1\sigma_2\sigma_1)^{-1}$$

Suppose the representation of this as a 3×3 matrix, using the Budapest switch augmented by t , is M . Then the representation matrix of the module is $M - id$. The nine codimension 1 subdeterminants are

$$p = 2t^{-2} + 5 + 2t^2, \quad (4 \text{ times}) \quad q = (2 + 2t^2)p(t^{-1}), \quad (\text{twice}) \quad q(t^{-1}), \quad p^2$$

This not only shows that K_3 is non-trivial but that it cannot be classical by 9.4.

10 Epilogue

Most calculations in this paper are done with Maple. At Andy Bartholomew's website at <http://www.layer8.co.uk/maths/> it is possible to download a C-program which calculates the invariants.

It is extremely unlikely that there are no non-trivial virtual knots for which these methods fail to distinguish it from the trivial knot. For example if the braid $\beta \in \overline{K}$ (see section 2), then the

closure of β possibly provides an infinite set of examples. However, to prove that an infinite set exists would require different methods.

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