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STATE MODELS AND THE JONES POLYNOMIAL

LOUIS H. KAUFFMAN

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§1. INTRODUCTION

IN THIS PAPER I construct a state model for the (original) Jones polynomial [5]. (In [6] a state model was constructed for the Conway polynomial.)

As we shall see, this model for the Jones polynomial arises as a normalization of a regular isotopy invariant of unoriented knots and links, called here the *bracket polynomial*, and denoted $\langle K \rangle$ for a link projection K. The concept of regular isotopy will be explained below. The bracket polynomial has a very simple state model.

In §2 (Theorem 2.10) I use the bracket polynomial to prove (via Proposition 2.9 and an observation of Kunio Murasugi) that the number of crossings in a connected, reduced alternating projection of a link L is a topological invariant of L. (A projection is reduced if it has no isthmus in the sense of Fig. 5.) In other words, *any two connected, reduced alternating projections of the link L have the same number of crossings*. This is a remarkable application of our technique. It solves affirmatively a conjecture going back to the knot tabulations of Tait, Kirkman and Little over a century ago (see [6], [9], [10]).

Along with this application to alternating links, we also use the bracket polynomial to obtain a necessary condition for an alternating reduced link diagram to be ambient isotopic to its mirror image (Theorem 3.1). One consequence of this theorem is that a reduced alternating diagram with twist number greater than or equal to one-third the number of crossings is necessarily chiral.

The paper is organized as follows. In §2 the bracket polynomial is developed, and its relationship with the Jones polynomial is explained. This provides a self-contained introduction to the Jones polynomial and to our techniques. The last part of §2 contains the applications to alternating knots, and to bounds on the minimal and maximal degrees of the polynomial. §3 contains the results about chirality of alternating knots. §4 discusses the structure of our state model in the case of braids. Here the states have an algebraic structure related to Jones's representation of the braid group into a Von Neumann Algebra.

§2. BRACKET INVARIANT

We first describe a general scheme of calculation from unoriented knot and link diagrams. This scheme associates a polynomial in three variables A, B and d to each diagram. It is well-defined on equivalence classes of diagrams. Two diagrams are *equivalent* if their underlying planar graphs are equivalent under orientation preserving homeomorphisms of the plane. Note that the Reidemeister moves change the graphical structure.

We distinguish three relations on diagrams: equivalence (as above), ambient isotopy and regular isotopy. Two diagrams are *ambient isotopic* if one can be obtained from the other by a sequence of Reidemeister moves of type I, type II and type III (see Fig. 1) plus equivalence as

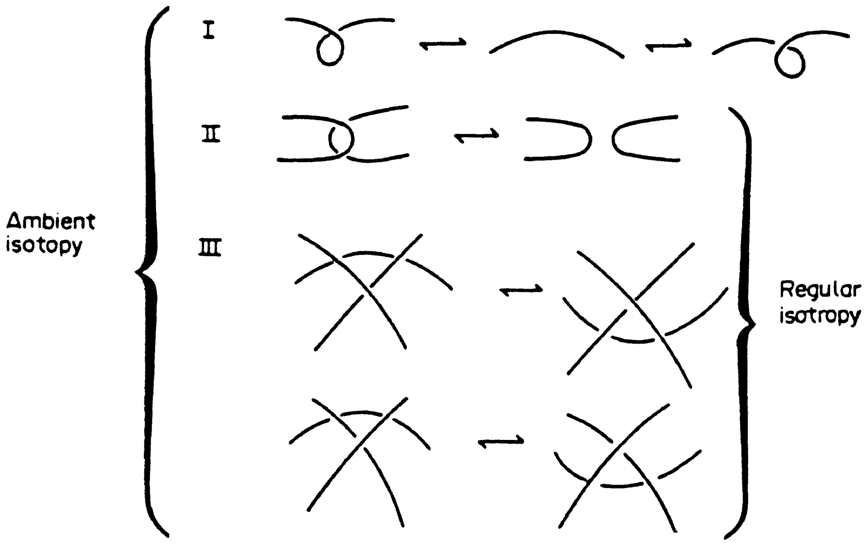


Fig. 1. Reidemeister moves

defined above. Two diagrams are *regularly isotopic* if they are *ambient isotopic without the use of the type I move*. Regular isotopy turns out to be a convenient concept for us.

DEFINITION 2.1. *Let K be an unoriented knot or link diagram. Let $\langle K \rangle$ be the element of the ring $\mathbb{Z}[A, B, d]$ defined by means of the rules*:

(i) $\langle 0 \rangle = 1$

(ii) $\langle 0 \cup K \rangle = d\langle K \rangle$, *$K$ not empty*

(iii) $\langle \times \rangle = A\langle \asymp \rangle + B\langle)(\rangle$.

Remark. A formula may involve the bracket and a few small diagrams. *These small figures represent larger diagrams that differ only as indicated in the small diagrams.*

The bracket, $\langle K \rangle$, is well defined on diagrams, but it is not invariant under any of the Reidemeister moves. It is the purpose of this section to determine relations among A, B, d so that $\langle K \rangle$ becomes invariant under Reidemeister moves. We will obtain invariance under types II and III, hence the use of regular isotopy.

Some comments are in order about the rules: Rule (i) says that $\langle K \rangle$ takes the value 1 on a single unknotted circle diagram. Rule (ii) says that $\langle K \rangle$ is multiplied by d in the presence of a disjoint circular component. This component can surround other parts of the diagram. Rule (iii) applies to diagrams that differ locally at the site of a single crossing. We can use rule (iii) to keep expanding the formulas until we reach diagrams consisting of disjoint unions of circles (Jordan curves in the plane). Rules (ii) and (iii) then imply that the value of $\langle K \rangle$ on a disjoint collection of circles is d raised to one less than the cardinality of the collection.

Note that rule (iii) entails the formula

$$\langle \times \rangle = B\langle \asymp \rangle + A\langle)(\rangle.$$

In fact, we can create a mnemonic for this expansion by labelling the crossings as shown in Fig. 2. *This label A marks the two local regions swept out by turning the overcrossing line counterclockwise until it coincides with the undercrossing line.*

For the expansion formula (iii) we can indicate which way a crossing is to be split by scoring a *marker* on it that connects the two regions that will be joined by the splitting. See Fig. 2.

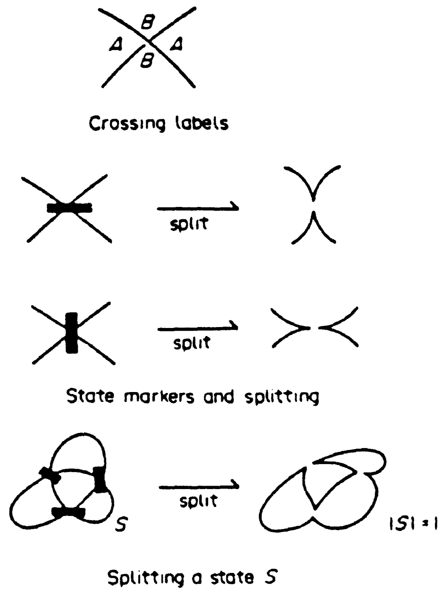


Fig. 2.

If U is the underlying planar graph for K, then a *state of* U is a choice of splitting marker for every vertex of U. Again see Fig. 2. I choose to call the underlying planar graph for a diagram K the *universe* for K (see [6]). This terminology distinguishes the underlying planar graph from the link projection and from other graphs that can arise. Thus we speak of the states of a universe.

Since splitting all the vertices of a state results in a configuration of disjoint circles, we see that the states are in one-to-one correspondence with final configurations in the expansion of the bracket. Accordingly, we define $\langle K \mid S\rangle$ for a diagram K and a state S by the formula

$$\langle K \mid S\rangle = A^i B^j$$

where i is the number of state markers touching A labels, and j is the number of state markers touching B labels. The total contribution of a given state to the polynomial is then given by the formula.

$$\langle K \mid S\rangle d^{|S|-1}$$

where $|S|$ denotes the number of circles in the splitting of S. View Fig. 2.

These observations are summarized in the statement of the following proposition, whose proof we omit.

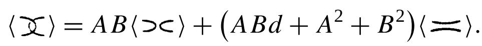
PROPOSITION 2.2. *$\langle K\rangle$ is uniquely determined on diagrams by the rules* (i), (ii), (iii). *It is given by the formula*

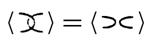
$$\langle K\rangle = \sum_S \langle K \mid S\rangle d^{|S|-1}$$

where this summation is over all states of the diagram, and S denotes the number of components in the splitting of a state S.

We now see how $\langle K\rangle$ behaves under elementary diagram moves, and consequently determine how to adjust A, B, and d to obtain a topological invariant.

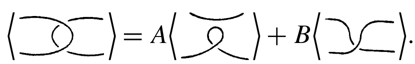
LEMMA 2.3. *The following formula holds, where the three diagrams represent the same projection except in the area indicated.*



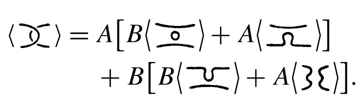
Hence  *for all diagrams if*

$$AB = 1 \quad \textit{and} \quad d = -A^2 - A^{-2}.$$

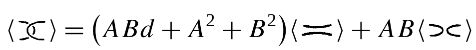
Proof.



Thus



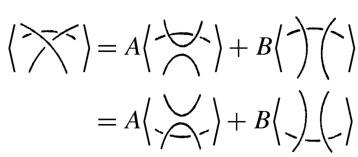
Hence



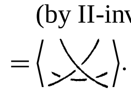
This completes the proof.

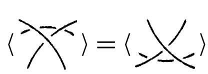
LEMMA 2.4. *Type II invariance for* $\langle\ \rangle$ *implies type III invariance.*

Proof.



(by II-invariance)



Hence . This is type III invariance.

Thus we see that by choosing

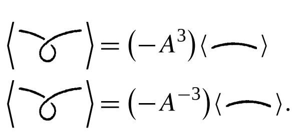
$$B = A^{-1},\ d = -A^2 - A^{-2}$$

$\langle K \rangle$ becomes a Laurent polynomial in A, and it is an invariant of regular isotopy (i.e. invariant under moves of type II and III). It is not invariant under the type I moves, but behaves as follows:

PROPOSITION 2.5. *With*

$$B = A^{-1},\ d = -A^2 - A^{-2}$$

then



Proof. This is a direct calculation, and is omitted.

From now on, unless otherwise specified, we assume that A, B and d *are chosen as indicated in Proposition* 2.5. With these choices, the bracket polynomial is an invariant of regular isotopy for unoriented knots and links.

The simplest invariant of regular isotopy for oriented diagrams is the *twist number* (or writhe) $w(K)$. This is the sum of the signs of all the crossings where each crossing is given a sign of plus or minus 1 according to the conventions shown in Fig. 3.

To obtain an invariant of ambient isotopy for oriented knots and links we define a Laurent polynomial $f[K]$ by the formula

$$f[K] = (-A)^{-3w(K)}\langle K\rangle,$$

where $w(K)$ denotes the twist number of the diagram K. The bracket is defined on oriented diagrams by forgetting the orientation.

THEOREM 2.6. *The polynomial* $f[K] \in Z[A, A^{-1}]$ *defined above is an ambient isotopy invariant for oriented links* K.

Proof. By combining the behaviour of the twist number under type I Reidemeister moves with the behaviour of the bracket (Proposition 2.5), it follows that $f[K]$ is invariant under type I moves. Thus $f[K]$ is invariant under all three moves, and is therefore an invariant of ambient isotopy.

Both the bracket and the polynomial $f[K]$ behave as follows on taking mirror images:

PROPOSITION 2.7. *Let* $K!$ *denote the mirror image of* K (*obtained by reversing all the crossings of* K). *Then*

$$\langle K!\rangle(A) = \langle K\rangle\left(A^{-1}\right)$$
$$f[K](A) = f[K]\left(A^{-1}\right).$$

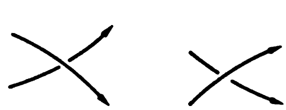
Proof. Just note that switching all crossings results in the replacement of every appearance of A by its inverse in the expansion of the bracket. This proves the first part. Since the twist number of a mirror image is the negative of the twist number of the original, the second part follows as well.

The Jones polynomial

Now recall that the Jones polynomial [5] is defined by the identities:

$$V_{\bigcirc} = 1$$
$$t^{-1}V_{\times} - tV_{\times} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)V_{\asymp}.$$

The Jones polynomial is an ambient isotopy invariant of oriented knots and links.



$\varepsilon = +1$ $\varepsilon = -1$

Fig. 3. Sign conventions

THEOREM 2.8.

$$V_K(t) = f[K]\big(t^{-1/4}\big).$$

Proof.

$$\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle)(\rangle$$
$$\langle \times \rangle = A^{-1}\langle \asymp \rangle + A\langle)(\rangle$$

Hence

$$A\langle \times \rangle - A^{-1}\langle \times \rangle = \big(A^2 - A^{-2}\big)\langle \asymp \rangle.$$

Multiplying this last formula by appropriate writhes, we have

$$A^4 f[\times] - A^{-4} f[\times] = \big(A^{-2} - A^2\big) f[\asymp].$$

The result follows at once by substituting t raised to the negative one-quarter power for A.

Figure 4 illustrates the calculation of the bracket for the Hopf link and for the trefoil knot. This calculation, in conjunction with Proposition 2.7, produces a short elementary proof of the distinction between the trefoil and its mirror image.

Remark. Some formal results about the Jones polynomial (reversing formula [11], [12], Birman "infinity"-formula [21]) follow immediately and trivially from Theorem 2.8 and the definition of the bracket polynomial. We leave the verification of these relations as an exercise for the interested reader.

Alternating links

We now give applications to alternating links. The first result determines the highest and lowest degree terms in the bracket polynomial for an alternating diagram.

PROPOSITION 2.9. *Let K be an alternating knot or link diagram that is connected and reduced. Let K be shaded so that all the crossings are of shaded type A (the regions labelled "A" are shaded). Then the term in $\langle K\rangle$ of highest degree in A has degree $V + 2W - 2$, where V is the number of crossings in K, and W is the number of white regions for this shading. The co-efficient of this power of A in $\langle K\rangle$ is $(-1)^{W-1}$.*

With the same hypotheses, the lowest degree term has degree $-V - 2(B-1)$, where B denotes the number of black (shaded) regions in the diagram. This term is monic with co-efficient $(-1)^{B-1}$.

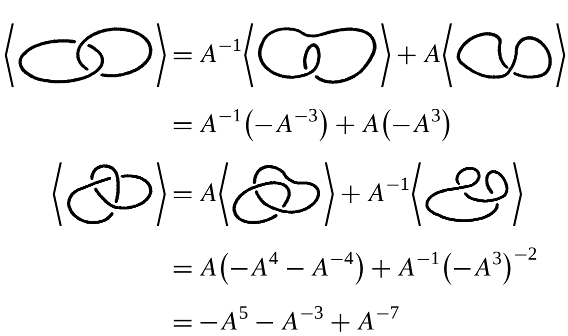


Fig. 4. Sample bracket calculations

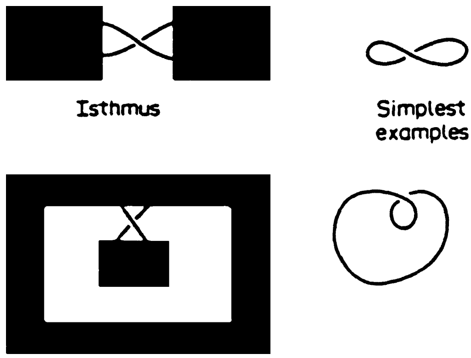


Fig. 5.

Before proving this result, some commentary on terminology is needed. A *reduced* diagram is one that does not contain an isthmus as shown in Fig. 5. An *isthmus* is a crossing in the diagram so that two of the four local regions at the crossing are part of the same region in the larger diagram.

Note that *a connected alternating diagram, when shaded in checkerboard fashion, has all of its crossings of the same shaded type* (see Fig. 6). By flipping the shading, if necessary, we may assume that it is the A-labelled regions that are shaded.

We shall call a diagram satisfying these hypotheses (connected, reduced) a *simple diagram*. Thus Proposition 2.9 says that if K is a simple diagram, then

$$\max \deg\langle K\rangle = V + 2W - 2$$
$$\min \deg\langle K\rangle = -V - 2B + 2.$$

The idea behind our pinpointing of the maximal degree is this: by choosing the state obtained by splitting every crossing in the A-direction, we obtain W components (where W is the number of white regions), and hence a degree of $V + 2(W - 1)$ from the corresponding part of the summation for the bracket polynomial.

Proof of 2.9. Let S be the state obtained by splitting every crossing in the diagram in the A-direction. Then $\langle K \mid S\rangle = A^V$, and $|S| = W$, where W is the number of white regions in the shading. Thus this state contributes the term

$$\langle K \mid S\rangle d^{|S|-1} = A^V d^{W-1}$$

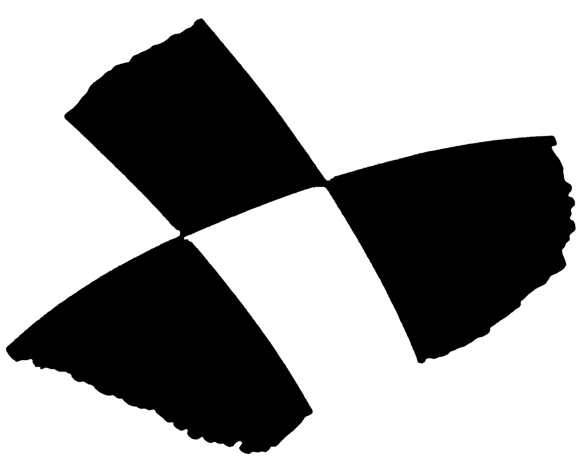


Fig. 6. Type A shading on an alternating projection.

to the state expansion of $\langle S\rangle$. Since

$$d = -A^2 - A^{-2}$$

this means that the highest degree contribution of the state S is the degree $V + 2(W - 1)$.

Now consider any other state S'. The state S' can be obtained from S by switching some subset of state markers of S. Thus there is a sequence of states $S(0), S(1), S(2), \ldots, S(n)$ so that $S = S(0)$, $S' = S(1)$, and $S(i+1)$ is obtained from $S(i)$ by switching one state marker from type A to type A^{-1}. Since a state marker of type A^{-1} contributes $(1/A)$, we see that $\langle K \mid S(i+1)\rangle = A^{-2}\langle K \mid S(i)\rangle$. Also, $|S(i+1)|$ is within 1 of $|S(i)|$, since switching a single state marker can change the component count of the split state by at most one. It follows that the maximal degree contribution of $S(i+1)$ is less than or equal to the maximal degree contribution of $S(i)$.

However, we assert that the maximal degree contribution actually falls from $S(0)$ to $S(1)$. This follows from the above assumption, if it is shown that *switching any state marker in S will cause a decrease in the number of components of the corresponding split state*. This follows from the assumption of diagram simplicity. (If the number of split components did not decrease from $S = S(0)$ to $S(1)$, then some white region would touch both sides of a crossing. This can only happen in the presence of an isthmus.)

Thus we have shown that the term of maximal degree in the entire bracket polynomial is contributed by the state S, and is not cancelled by terms from any other state. This completes the proof.

Finally we give the main application.

THEOREM 2.10. *The number of crossings in a simple alternating projection of a link L is a topological invariant of L. Hence any two simple alternating projections of a given link have the same number of crossings.*

Proof. Let *span*(K) denote the difference

$$\operatorname{span}(K) = \operatorname{maxdeg}\langle K\rangle - \operatorname{mindeg}\langle K\rangle.$$

Then

$$\begin{aligned}\operatorname{span}(K) &= [V + 2W - 2] - [-V - 2B + 2]\\ &= 2V + 2(W + B) - 4.\end{aligned}$$

Since $W + B$ equals the total number of regions in the diagram, and this exceeds the number of crossings by two, we have

$$\operatorname{span}(K) = 2y + 2(V + 2) - 4 = 4V.$$

This completes the proof.

Remark. It is worth remarking that Proposition 2.9 can be generalized to imply an inequality:

$$\operatorname{span}(K) \leqslant 4V,$$

for an arbitrary (not necessarily alternating) diagram K.

This result has been observed by Kunio Murasugi and (independently) Morwen Thistlethwaite (see [14] and [17]), each using the ideas of an early version of this paper. We give here our short proof via the:

DUAL STATE LEMMA 2.11. *Let S be a state for a connected universe U. Let $\widehat{S}$ denote the state obtained from S by reversing all the state markers of S (call this the dual state of S). Then*

$$|S| + |\widehat{S}| \leqslant R$$

where R denotes the number of regions in U.

Proof. The proof is by induction on the number of vertices V in U. It is easily seen to be true for $V = 0, 1, 2$. Therefore suppose the result true for all universes with less than V vertices. Let U be connected, with V vertices. Let U' and U'' be the two universes obtained by splitting U at a given vertex P in the two possible ways. Then, by connectivity of U, one of U' or U'' is also connected. We may suppose that U' is connected. Apply the induction hypothesis to U'.

If S is a state of U, then either S or $\widehat{S}$ is split at the vertex P in the same direction that formed U'. We can assume that S is so split. Then, by ignoring the site at P, S can be construed as a state S' of U'.

By induction, $|S'| + |\widehat{S'}| \leqslant R'$ where R' denotes the number of regions of U'. By construction, $R' = R - 1$ where R is the number of regions of U. And S' and S have the same number of split components: $|S| = |S'|$. On the other hand, it is possible that $\widehat{S}$, being obtained from $\widehat{S'}$ by splicing at the site p, may have (at most) one less split component than $\widehat{S'}$. Thus $|\widehat{S'}| + 1 \geqslant \widehat{S}$. These facts imply the inequality $|S| + |\widehat{S}| \leqslant R$, completing the inductive proof of the Lemma.

The inequality mentioned prior to the proof of this lemma now follows by repeating the proof of 2.9, using the state S where all the markers are of type A. No relation with the checkerboard shading is required, nor do we need assume anything other than connectivity of the diagram. Wu then obtain:

$$\begin{aligned} \text{maxdeg}\langle K\rangle &\leqslant V + 2(|S| - 1) \\ \text{mindeg}\langle K\rangle &\geqslant -V - 2(|\widehat{S}| - 1). \end{aligned}$$

The lemma, in conjunction with the calculation of 2.10 then gives the inequality: $\text{span}(K) \leqslant 4V$.

Finally we note that both Murasugi and Thistlethwaite prove the stronger inequality: $\text{span}(K) < 4V$ when K is a connected non-alternating diagram. Wu [18] gives a proof of this inequality by strengthening our dual state Lemma.

§3. CHIRALITY OF ALTERNATING LINKS

Here we apply Proposition 2.9 to obtain a necessary condition for an oriented alternating link to be achiral (ambient isotopic to its mirror image).

THEOREM 3.1. *Let K be a simple alternating diagram shaded as in Proposition* 2.9. *Let W and B denote the number of white and black regions in this shading. Suppose that K has twist number $w(K)$. If K is ambient isotopic to $K!$ then*

$$3w(K) = W - B.$$

It follows from this formula that

$$\begin{aligned} W &= (1/2)(V + 3w(K)) + 1 \\ B &= (1/2)(V - 3w(K)) + 1 \end{aligned}$$

where V denotes the number of crossings in the diagram.

Proof. We use the Jones polynomial in the form of the ambient isotopy invariant $f[K]\ (A)$ (see Theorem 2.8). It follows from 2.9 and the definition of $f[K]$ that $f[K]$ has maximal and minimal A-degrees *max*(f) and *min*(f) given by the formulas

$$\max(f) = -3w + V + 2(W-1)$$
$$\min(f) = -3w - V - 2(B-1)$$

where w is the twist number $w = w(K)$.

In order for K to be achiral it is necessary that $-\min(f) = \max(f)$ (by Proposition 2.6). Thus

$$3w + V + 2(B-1) = -3w + V + 2(W-1).$$

Hence

$$3w = W - B.$$

The remaining formulas follow by using the relation $W + B = V + 2$. This completes the proof.

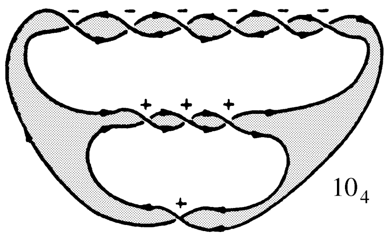
COROLLARY 3.2. *Let K be a simple alternating diagram. Let $T = |w(K)|$. Assume K is not the unknotted circle diagram. Then, if $T \geqslant V/3$ (V is the number of crossings in the diagram) then K is chiral.*

This corollary is an easy consequence of Theorem 3.1. We omit the proof. It follows at once from Corollary 3.2 that special alternating (simple) links are chiral. This fact was first proved by Murasugi [13], using his signature invariant.

Remark. There exist chiral alternating knots satisfying the condition $3w(K) = W - B$ of Theorem 3.1. Thus the condition of 3.1 is necessary but not sufficient for achirality. Figure 7 depicts such an example. A signature calculation shows that this knot (10_4 in Rolfsen's tables [15]) is chiral.

Note also that it is easy to produce an alternating prime knot of zero twist number that is chiral (remove two crossings from the top line of Fig. 7). Erica Flapan has observed the non-alternating knot 10_{125} to have the same property (chiral, twist number zero, prime).

Remark. It is entirely possible that the venerable conjecture (that the twist number is a topological invariant for reduced alternating diagrams) is true. This would generalize 3.1 except for the case of twist number zero. In this case, 3.1 demands $B = W$ for achirality. More may be true. We make the following conjecture.



$w = -2,\ W = 3,\ B = 9$

Fig. 7.

CONJECTURE. *Let K be a prime reduced alternating diagram, and suppose that K is achiral with twist number zero. Then the two planar graphs $B(K)$ and $W(K)$ are isomorphic as abstract graphs.*

Here $B(K)$ is the graph formed from the black shading: one vertex for each shaded region, one edge for each crossing shared by shared regions. $W(K)$ is the planar dual to $B(K)$ formed from the white regions. (See Note added in Proof.)

§4. BRAID STATES AND THE DIAGRAM ALGEBRA

The bracket also provides an entry into the representation theory associated with the Jones polynomial. In order to see this we define a *diagram algebra*, $D[n]$, based on the patterns shown in Fig. 8. Here diagrams with free ends are multiplied as braids, while multiplication by the closed loop δ denotes disjoint union. Addition is formal with no imposed relations. As the figure shows, the resulting (multiplicative) monoid has relations

$$\begin{cases} h_i^2 = \delta h_i = h_i \delta \\ h_i h_{i+1} h_i = h_i \\ h_{i+1} h_i h_{i+1} = h_{i+1} \\ h_i h_j = h_j h_i, \quad |i-j| > 1. \end{cases}$$

(We omit the proof here that it has exactly these relations. See [7], [8].)

The original Jones polynomial was defined via a representation into an abstract algebra satisfying (essentially) these multiplicative relations. The diagram algebra forms the beginning of a direct geometric connection between these algebras and the theory of braids. In our terms the relation $\langle \asymp\!\!\!\!\times \rangle = A\langle \asymp \rangle + A^{-1}\langle\rangle\langle\rangle$ becomes the pattern for a representation of the braid group (see [5]) into the free additive algebra with the above multiplicative relations and

$$\delta = d = -A^2 - A^{-2}$$

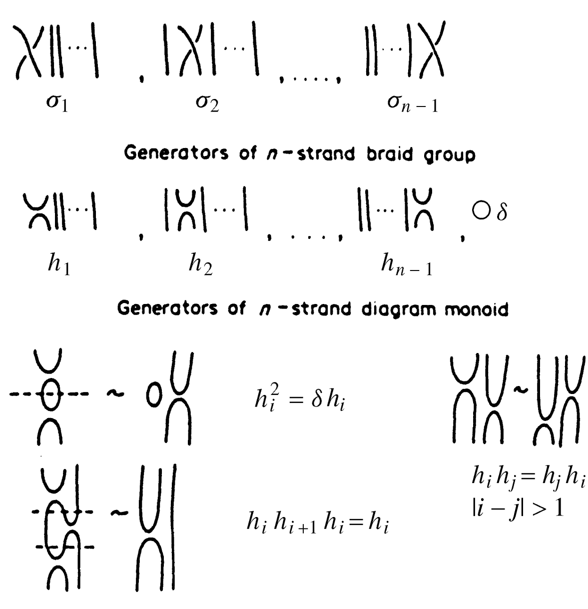
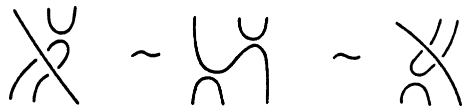


Fig. 8.



$$h_{i+1}\sigma_i\sigma_{i+1} = h_{i+1}h_i = \sigma_i\sigma_{i+1}h_i$$

Fig. 9. A basic braid monoid relation

(specializing the loop variable). If σ_i is the ith braid generator then the representation is given by the formula

$$\rho(\sigma_i) = Ah_i + A^{-1}1.$$

Let $B[n]$ denote the n-strand braid group. A braid is a product of the generators

$$\sigma_1^{\pm 1}, \sigma_2^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}.$$

(See Fig. 8.) For a braid b, let $\langle b \rangle$ denote the evaluation of the bracket polynomial on the closure of b.

Since the states corresponding to a given braid b are obtained by eliminating crossings horizontally $\langle \asymp \rangle$ or vertically $\langle)(\rangle$, we see that the generators for the diagram monoid correspond to horizontal splits on the generators of the braid group. (Vertical splits give identity braids.)

The relations in the diagram monoid allow one to algebraically determine the number of components in the closure of h—hence the value of $\langle h \rangle$, for any product h in $D[n]$. This gives a diagrammatic interpretation to trace computations in the representation theory.

Note how the bracket expansion and the form of the representation fit together. By representing each braid generator as a sum of two algebraic terms, the product corresponding to a braid word has a power-of-two number of terms. Each term is a power of A multiplied by a product of generators of the diagram algebra. Each such product corresponds to one of the states in the bracket expansion.

Finally it is worth mentioning that a rich generalization of the braid group is obtained by allowing products of braid generators with elements of the diagram monoid. The resulting system, up to regular isotopy, can be defined so that it has relations corresponding to standard braiding relations, diagram monoid relations, plus extra relations of the type illustrated in Fig. 9. Call this structure the *Braid Monoid*. (See [7] and [8].) Yetter [19] has studied a version of the braid monoid, and Birman and Wenzel [4] use an algebra derived from it to study representations of the braid group and the Kauffman polynomial [8].

This section has been an introduction to braid states and the diagram algebra. For relations with chromatic polynomials and the Potts model ([1], [5]), see [7]. (By associating an appropriate alternating link diagram to any planar graph, and by choosing A, B, d appropriately, dichromatic polynomials and partition functions (for the Potts model) can be computed using the bracket expansion. Thus our formalism provides a relationship among knot theory, graph theory and physics. In the case of braids, the simple topology of the diagram algebra underlies all three aspects.)

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Note added in Proof

The invariance of the twist number for reduced alternating diagrams has been proved (independently) by Murasugi and Thistlethwaite.

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Department of Mathematics, Statistics and Computer Science
The University of Illinois at Chicago
Chicago, IL 60680, U.S.A.